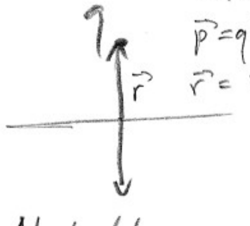
Dipolstrahlung

klassische Erzeugung Strahlung: Dipol

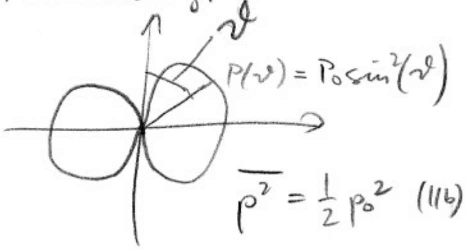


Dipol Moment
$\vec{p} = q\vec{r}$
$\vec{r} = \vec{r}_0 \sin(\omega t)$ (10)

Abstrahlung über alle Winkel ϑ gemittelt :
$$\overline{P} = \frac{2}{3} \frac{\overline{p^2}\,\omega^4}{4\pi\varepsilon_0 c^3} \quad (11)$$

(Annahme $\lambda \gg |\vec{r}_0|$, Dipolnäherung)

Abstrahlung:



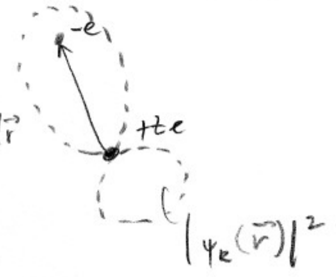
$\overline{p^2} = \frac{1}{2} p_0^2$ (11b)

quantenmechanisch : $\overline{p^2} \longrightarrow \langle p^2 \rangle$

$\overline{p^2}$: klassischer Mittelwert

$\langle p^2 \rangle$: qm Erwartungswert des el. Dipolmomentes

$\langle k_k \rangle = \int \psi_k^* \vec{r} \psi_k d\vec{r}$



$$\langle p \rangle = e \langle r \rangle = e \int d\vec{r}\; \psi_i^*(\vec{r})\, \vec{r}\, \psi_k(\vec{r}) \quad (12)$$

$\psi_\alpha(\vec{r})$ Wellenfunktion zum Zustand mit Quantenzahlen $\alpha = (n_\alpha, l_\alpha, m_{e\alpha}, m_{s\alpha})$
($\alpha = i$ und k) (z.B. H-Atom)

$\int d\vec{r}$ integration über Raumkoordinaten Elektron

$d\vec{r} = dx\,dy\,dz$ oder $dr = r^2 dr \sin\vartheta\, d\vartheta\, d\varphi$

(13) $\langle p \rangle =: \boxed{M_{ik} = e \int d\vec{r}\, \psi_i^* \, \vec{r}\, \psi_k}$

hängt von i,k ab: Matrix element „Übergangsmatrixelement"

wobei $M_{ik} = M_{ki}^*$
$|M_{ik}| = |M_{ki}|$
$\Rightarrow B_{ik} = B_{ki}$

ersetze $\overline{p^2} = \frac{1}{2}(p_0)^2 \longrightarrow 2\,|M_{ik}|^2$ in (11)

$$\Rightarrow \quad \overline{P} = \frac{4}{3} \frac{\omega_{ik}^4}{4\pi\varepsilon_0 c^3} |M_{ik}|^2 = \text{mittlere Leistung / Atom} \quad (15)$$

gemäß Definition von A_{ik} via mittlere emittierte Leistung $= A_{ik} h\nu_{ik}$ (16)

$$\boxed{A_{ik} = \frac{2}{3} \frac{\omega_{ik}^3}{\varepsilon_0 c^3 h} |M_{ik}|^2} \quad (17)$$

Einstein-Koeffizient Spontane Emission

Übergangswahrscheinlichkeit pro Sekunde

unabhängig von Strahlungsfeld $w(\nu)$

quantenmechanische Motivation der Dipolübergangsmatrixelemente

$H\psi = E\psi$ stat. Schr. gl. / Eigenwert Problem

$H\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$ zeitabh. Schr. gl.

Nehme $H = H_0 + V'(x,t)$

$\uparrow$ Störpotential z.B. $V' = eAx\cos(\omega t)$

$F_x = -\frac{\partial V'}{\partial x}$

$H_0 = -\frac{\hbar^2}{2m}\Delta + V(x)$ ungest. Problem

$H_0 u_\alpha = E_\alpha u_\alpha$

$H_0 u_\beta = E_\beta u_\beta$

$\psi_{\alpha,\beta} = u_{\alpha,\beta}(x,y,z)\cdot e^{-\frac{i}{\hbar}E_{\alpha,\beta}t}$ Lsg. zeitabh. Schr.

Ansatz: $\Psi = \underset{c_\alpha(t)}{\underset{\shortparallel}{c_\alpha}}\,\psi_\alpha + c_\beta\psi_\beta$ auch Lsg. von $H_0\Psi = -\frac{\hbar}{i}\frac{\partial \Psi}{\partial t}$ aber nicht von H

finde Näherungslösung $\rightarrow$ Übung $\overset{*}{\Rightarrow}$

* unter Annahme:
- ebene Welle
- nur $\vec{E}$-feld (nicht $\vec{B}$)
- $\vec{E}$ feld homogen $(\lambda \gg a_0)$
- Übergang von $\psi_\alpha \rightarrow \psi_\beta$ $\leftarrow$ stat. Endzustand

$\uparrow$ stat. Anfangsz.

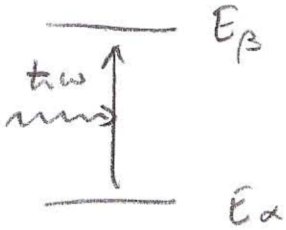
$\omega_{\beta\alpha} := \frac{E_\beta - E_\alpha}{\hbar}$

$$\frac{dc_\beta}{dt} \cong \frac{i}{\hbar} e^{\frac{i}{\hbar}(E_\beta - E_\alpha)t} E_0 \cos(\omega t) M_{\beta\alpha} \qquad M_{\beta\alpha} = \int d\vec{r}\, u_\beta^* x\, u_\alpha$$

$$= \frac{i}{2\hbar} E_0 M_{\beta\alpha}\left(e^{i(\omega_{\beta\alpha}+\omega)t} + e^{i(\omega_{\beta\alpha}-\omega)t}\right)$$

$$c_\beta(t) = \int_0^t \frac{dc_\beta}{dt}\,dt = \frac{E_0}{2\hbar} M_{\beta\alpha}\left\{\frac{e^{i(\omega_{\alpha\beta}+\omega)t}-1}{(\omega_{\alpha\beta}+\omega)} + \frac{e^{i(\omega_{\beta\alpha}-\omega)t}-1}{(\omega_{\beta\alpha}-\omega)}\right\}$$

Absorption



$$\omega_{\beta\alpha} = \frac{E_\beta - E_\alpha}{\hbar} > 0$$

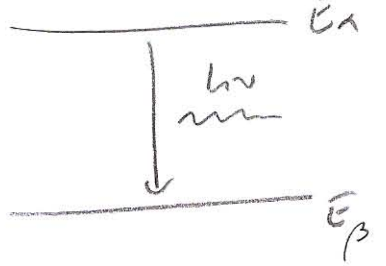
$$\rightarrow \quad |\omega_{\beta\alpha} + \omega| \gg |\omega_{\beta\alpha} - \omega| = \Delta\omega$$

$$c_\beta(t) \cong \frac{E_0}{2\hbar} M_{\beta\alpha} \frac{e^{i(\omega_{\beta\alpha} - \omega)t} - 1}{\omega_{\beta\alpha} - \omega}$$

calculate $c_\beta^* c_\beta$, use $(e^{i\alpha} - 1)(e^{-i\alpha} - 1)$
$= 2(1 - \cos(\alpha)) = 4\sin^2\left(\frac{\alpha}{2}\right)$

$$|c_\beta(t)|^2 \sim \frac{E_0^2}{\hbar^2} |M_{\beta\alpha}|^2 \frac{\sin^2\left(\frac{1}{2}\Delta\omega t\right)}{(\Delta\omega)^2}$$

Emission

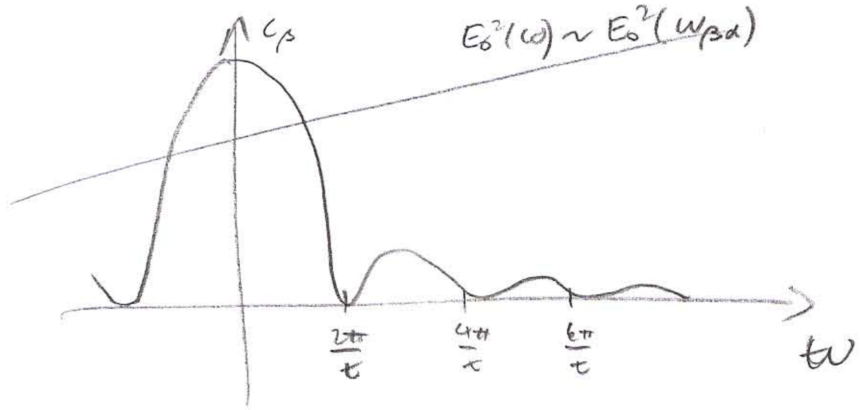


$$\omega_{\beta\alpha} = \frac{E_\beta - E_\alpha}{\hbar} < 0$$

$$|\omega_{\beta\alpha} - \omega| \gg |\omega_{\beta\alpha} + \omega| = \Delta\omega$$

ditto Absorption, i.e. same $|c_\beta(t)|^2$

$\rightarrow$ $\boxed{\text{Absorption} = \text{Emission } W_i}$



$$\int_{-\infty}^{\infty} d\Delta\omega \, \frac{E_0^2(\omega_{\beta\alpha}) |M_{\beta\alpha}|^2}{\hbar^2} \frac{\sin^2\left(\frac{1}{2}\Delta\omega\right)}{\Delta\omega^2}$$

$$= \frac{E_0^2(\omega_{\beta\alpha})\,|M_{\beta\alpha}|^2}{\hbar^2}\,\frac{t^2}{4}\int_{-\infty}^{\infty}\frac{\sin^2(\frac{1}{2}\Delta\omega t)}{(\frac{1}{2}\Delta\omega t)^2}\,d(\Delta\omega)$$

$$= \frac{E_0^2(\omega_{\beta\alpha})\,|M_{\beta\alpha}|^2}{\hbar^2}\,2t\,\underbrace{\int_{-\infty}^{\infty}\frac{\sin^2\xi}{\xi^2}\,d\xi}_{\pi} \qquad \frac{1}{2}\Delta\omega t = \xi, \quad d\Delta\omega = \frac{2}{t}\,d\xi$$

$$= \frac{\pi E_0^2(\omega_{\beta\alpha})\,|M_{\beta\alpha}|^2}{2\hbar^2}\,t$$

Übergangsw. wächst mit t

$$W_{\alpha\beta} = \frac{|c_\beta(t)|^2}{t} = \frac{\pi E_0^2(\omega_{\beta\alpha})\,|M_{\beta\alpha}|^2}{2\hbar^2}$$

$w(\omega_{\beta\alpha})$

thermisches Ggw, isotrop, addiere x, y, z Komp.

$$B_{\beta\alpha} = \frac{\pi}{3\varepsilon_0\hbar^2}\,|M_{\beta\alpha}|^2 \qquad \text{Abs. / stim. Em.}$$

Herleitung Einstein Koeffizienten

thermisches Strahlungsfeld: inkohärent → addiere Intensitäten / Übergangsraten für einzelne x, y, z Komponenten von $\vec{E}$.

polarisiert $\parallel x$, fortschreitend $\parallel z$, $E_x = B_y$, E-dichte:

$$\frac{1}{8\pi}\left(E_x^2 + B_y^2\right) = \frac{1}{4\pi} E_x^2$$

$$E_x = E_0 \cos(\omega t)$$

$$\overline{E_x^2} = \frac{1}{2} E_{0x}^2 \quad \text{zeitmittel}$$

$$w_x(\omega)\, d\omega = \frac{1}{4\pi} E_x^2(\omega)\, d\omega = \frac{1}{8\pi} E_{0x}^2\, d\omega$$

aber im isotropen Feld: $w_x = w_y = w_z = \frac{1}{3} w \to E_{0x}^2 = E_{0y}^2 = E_{0z}^2 = \frac{8\pi}{3} w(\omega)$

$$\to W_{\alpha\beta} = W_{\alpha\beta}^x + W_{\alpha\beta}^y + W_{\alpha\beta}^z = \overbrace{\frac{4\pi^2}{3\hbar^2} \underbrace{|M_{\beta\alpha}|^2}_{= |M_{\beta\alpha}^x|^2 + |M_{\beta\alpha}^y|^2 + |M_{\beta\alpha}^z|^2}}^{B_{\beta\alpha}} w(\omega_{\beta\alpha}) \xrightarrow[SI]{} \frac{\pi}{3\varepsilon_0\hbar^2} |M_{\beta\alpha}|^2\, w(\omega_{\beta\alpha})$$

$$\boxed{B_{\beta\alpha} = \frac{\pi}{3\varepsilon_0\hbar^2} |M_{\beta\alpha}|^2}$$

benutze A_{ik} (17)

$$\frac{A_{ik}}{B_{ik}} = \frac{\cancel{8}}{\cancel{3}} \frac{\omega^3}{\cancel{4} c^3 \cancel{\hbar}} \cdot \frac{\cancel{3}\hbar}{\cancel{8}\pi^2}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} = w(\omega) \to \frac{8\pi h\nu^3}{c^3} \quad \text{dito vorher}$$

$$w(\omega)\, d\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\, d\omega = \frac{8\pi h\nu^3}{c^3}\, d\nu = w(\nu)\, d\nu$$