SUPPLEMTARY NOTE 1: WEAK LOCALIZATION FORMALISM

Here we outline the main steps of the algorithm leading to the Cooperon operator, $\mathcal{H}$ following the standard WL formalism [1] starting from the single particle Hamiltonian, $H_{\mathbf{p}}$, (Eqs. (5) and (1), in the main manuscript, respectively).

The impurity mediated Cooperon satisfies,

$$C_{\mathbf{p},\mathbf{p}'}(\mathbf{q}) = |V_{\mathbf{p},\mathbf{p}'}|^2 + \sum_{\mathbf{p}''} |V_{\mathbf{p},\mathbf{p}''}|^2 G^{+}_{-\mathbf{p}''+\hbar\mathbf{q},\epsilon+\hbar\omega} G^{-}_{\mathbf{p}'',\epsilon} C_{\mathbf{p}'',\mathbf{p}'} \ . \tag{1}$$

The propagation of the particles is described by impurity averaged advanced (A) and retarded (R) Green's functions, given by

$$G^{\pm}(\mathbf{p},\epsilon) = \frac{1}{\epsilon - H_{\mathbf{p}} \pm i\frac{\hbar}{2\tau_0}} \ . \tag{2}$$

τ_0 is the impurity scattering relaxation time, $D = v^2\tau_1/2$ is the diffusion coefficient in two dimensions expressed as a function of the transport time, τ_1.

The Cooperon operator is obtained by linearizing Eq. (1) in an iterative approach [1, 3–5] leading to a formal equation written as,[1]

$$C_{\mathbf{p},\mathbf{p}'}(\mathbf{q}) = \frac{|V_{\mathbf{p},\mathbf{p}'}|^2}{\tau_0 \mathcal{H}_0} \ , \tag{3}$$

where $\mathcal{H}_0$ is an operator in the 4-dim total spin space,

$$\begin{aligned} \mathcal{H}_0 = Dq^2 + \frac{1}{\tau_\varphi} + 2k_F^2\left[(\alpha+\beta)^2\tau_1 + \beta_3^2\tau_3\right] J_z^2 + 2k_F^2\left[(\alpha-\beta)^2\tau_1 + \beta_3^2\tau_3\right] J_x^2 \\ + 2k_F(\alpha-\beta)\tau_1 v q_z J_x - 2k_F(\alpha+\beta)\tau_1 v q_x J_z \ . \end{aligned} \tag{4}$$

$-i\omega$ is replaced by $1/\tau_\varphi$, the dephasing time, a descriptor of the inelasticity of the propagation. We define,

$$
\begin{aligned}
Q_S &= \frac{2m^*(\alpha+\beta)}{\hbar} , \\
Q_A &= \frac{2(\alpha+\beta)}{\hbar} , \\
Q_3 &= \frac{2m^*\beta_3}{\hbar}\sqrt{\frac{\tau_3}{\tau_1}} ,
\end{aligned}
\tag{5}
$$

and rewrite $\mathcal{H}_0$ as

$$
\mathcal{H}_0 = Dq^2 + \frac{1}{\tau_\varphi} + D\left\{\left[Q_S^2 + Q_3^2\right] J_z^2 + \left[Q_A^2 + Q_3^2\right] J_x^2 + 2Q_A q_z J_x - 2Q_S q_x J_z\right\} . \tag{6}
$$

CORRECTIONS TO THE MAGNETOCONDUCTIVITY

In the presence of a quantizing magnetic field, the position representation of the Green's function $G^{\pm}(\mathbf{r},\mathbf{r}')$ is modified as [6]

$$
G(\mathbf{r},\mathbf{r}') = e^{\frac{ie}{\hbar}\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(\mathbf{l})\cdot d\mathbf{l}} G^{\pm}(\mathbf{r},\mathbf{r}') . \tag{7}
$$

The change in the phase of the Green's function induces a transformation in the position representation of $\mathcal{H}(\mathbf{r},\mathbf{r}')$ which now satisfies an eigenfunction-eigenvalue equation,

$$
\int e^{i\frac{2e}{\hbar}\mathbf{A}\cdot(\mathbf{r}'-\mathbf{r})}\mathcal{H}_0(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}' = \mathcal{E}\psi(\mathbf{r}) . \tag{8}
$$

The integral equation is linearized by expanding the integrant in a power series in $\mathbf{r}'-\mathbf{r}$ assumed to be small in comparison with the electron mean free path. In this approximation $\mathcal{H}_0(\mathbf{r},\mathbf{r}')$ satisfies,

$$
\left\{1 + \left(-i\nabla + \frac{2e}{\hbar}\mathbf{A}\right)\cdot\nabla_{\mathbf{q}} + \frac{1}{2}\left[\left(-i\nabla + \frac{2e}{\hbar}\mathbf{A}\right)\cdot\nabla_q\right]^2\right\}\mathcal{H}_0\,|_{\mathbf{q}=0}\,\psi(\mathbf{r}) = \mathcal{E}\psi(\mathbf{r}) . \tag{9}
$$

In a selection of axes with $\hat{y}$ perpendicular on the plane, the magnetic vector potential $\mathbf{A}$ in the Landau gauge is $\mathbf{A} = \{A_x = Bz, A_y = 0, A_z = 0\}$. Consequently, Eq. (9) is transformed into, with $\mathcal{H}_0(\mathbf{q})$ and its derivatives obtained from Eq.(6),

$$\left\{\frac{1}{\tau_\varphi} + D\left[Q_S^2 + Q_3^2\right] J_z^2 + \left[Q_A^2 + Q_3^2\right] J_x^2 - 2DQ_S J_z \left(-i\nabla_x + \frac{2eB}{\hbar} z\right) + 2DQ_A J_x \left(-i\nabla_z\right) \right.$$
$$\left. + D\left(-i\nabla_x + \frac{2eB}{\hbar} z\right)^2 + D\left(-i\nabla_z\right)^2 \right\} \psi(\mathbf{r}) = \mathcal{E}\psi(\mathbf{r}) \; . \tag{10}$$

With $z_0 = k_x\hbar/2eB$, we define canonical operators,

$$\begin{aligned} -i\nabla_z &= \sqrt{\tfrac{2eB}{\hbar}} \tfrac{(a-a^\dagger)}{i\sqrt{2}} \;, \\ z + z_0 &= \tfrac{1}{\sqrt{\frac{2eB}{\hbar}}} \tfrac{(a+a^\dagger)}{\sqrt{2}} \;, \end{aligned} \tag{11}$$

such that we obtain for the characteristic equation in the number representation,

$$\left\{\frac{1}{\tau_\varphi} + D\left(Q_S^2 + Q_3^2\right) J_z^2 + \left(Q_A^2 + Q_3^2\right) J_x^2 - DQ_S J_z \sqrt{\frac{4eB}{\hbar}}(a + a^\dagger) - iDQ_A J_x \sqrt{\frac{4eB}{\hbar}}(a - a^\dagger) \right.$$
$$\left. + D\left(\frac{4eB}{\hbar}\right)\left(a^\dagger a + \frac{1}{2}\right)\right\} |u\rangle = \mathcal{E}|u\rangle \; , \tag{12}$$

where $|u\rangle$ is the corresponding eigenket.

The left-hand side of the Eq. (12) defines operator $\mathcal{H}$, which maintains the structure of the original Cooperon, Eq. (6) in spin space, with q^2 being replaced by $\frac{4eB}{\hbar}\left(a^\dagger a + \frac{1}{2}\right)$, while its components q_x and q_z were replaced by $\sqrt{\frac{4eB}{\hbar}}(a+a^\dagger)/2$ and $\sqrt{\frac{4eB}{\hbar}}(a-a^\dagger)/2i$ respectively.

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