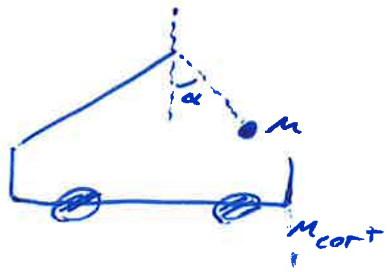
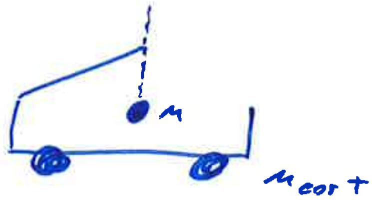
Cart with Pendulum

① 

$\vec{v} = 0$

$\vec{v}_{cart} = 0$

② 

$\vec{v} \neq 0$

$\vec{v}_{cart} \neq 0$

Conservation of Energy:

$$E_1 = E_2$$

$$E_{kin\,1} + E_{pot\,1} = E_{kin\,2} + E_{pot\,2}$$

$$0 + mg\ell(1-\cos\alpha) = \frac{1}{2}mv^2 + \frac{1}{2}m_{cart}v_{cart}^2$$

$$\longrightarrow \quad mg\ell(1-\cos\alpha) = \frac{1}{2}mv^2 + \frac{1}{2}m_{cart}v_{cart}^2$$

Conservation of Momentum:

$$\vec{p}_{tot\,1} = \vec{p}_{tot\,2}$$

$$0 = m\vec{v} + m_{cart}\vec{v}_{cart} \qquad \vec{v} \parallel \vec{v}_{cart}$$

$$\longrightarrow \quad v_{cart} = \frac{-m}{m_{cart}}v$$

Together :

$$mgl(1-\cos\alpha) = \frac{1}{2}mv^2 + \frac{1}{2}m_{cart}\frac{m^2}{m_{cart}^2}v^2$$

$$mgl(1-\cos\alpha) = \frac{1}{2}\left(m + \frac{m^2}{m_{carT}}\right)v^2$$

$$v^2 = \frac{2mgl(1-\cos\alpha)}{m + \frac{m^2}{m_{cart}}}$$

$$\boxed{v = \sqrt{\frac{2mgl(1-\cos\alpha)}{m + \frac{m^2}{m_{cart}}}}}$$

$m_{cart} >> m$, $v \to \sqrt{2gl(1-\cos\alpha)}$

$v_{cart} \to 0$

X $m_{cart} << m$ $v \to 0$

$v_{cart} \to -\infty$

$1 - \frac{m}{m_{cart}}$

board) **Conservation of (linear) momentum** , general treatment of collisions

reminder , momentum (linear) : $\vec{p} = m\vec{v}$

conservation of momentum : system with N objects (particles)

$$\vec{P}_{sys} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i$$

$$= M \cdot \vec{v}_{cm}$$

M: total mass ; $\vec{v}_{cm}$: velocity of the center of mass

(see e.g. Eq. 5.20 Tipler, Sec 5.5, p 149)

differentiate $\left(\frac{d}{dt}\right)$: $\frac{d\vec{P}_{sys}}{dt} = M \frac{d\vec{v}_{cm}}{dt} = M \cdot \vec{a}_{cm}$ ← acceleration

Newton's 2nd law $\left(\vec{F} = \frac{d\vec{p}}{dt}\right)$ $= \sum_i \vec{F}_{ext} = \vec{F}_{net\,ext}$

net (total) external force acting on the system

=> when $\sum \vec{F}_{ext} = 0$

$$\boxed{\vec{P}_{sys} = \sum_i m_i \vec{v}_i = \text{constant}}$$ conservation of momentum

simple example : 1D, central collision

(slide) (board)

energy conservation + momentum conservation %

$\boxed{Q}$: "additional" energy involved in the process ; e.g. : - thermal energy (dissipation, friction)
- mechanical energy (deformation)

(board) def, case $\underline{Q = 0}$: elastic collision (no energy "losses" for kinetic energy)

$\underline{Q > 0}$: inelastic collision

if $v_1' = v_2'$, fully inelastic

energy cons

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + Q$$

momentum cons.

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'$$

vector relation,
orientation,
ref system

① Elastischer Stoss

Energiesatz:

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

I) $$m_1 (v_1^2 - v_1'^2) = m_2 (v_2'^2 - v_2^2)$$

Impulssatz:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

II) $$m_1 (v_1 - v_1') = m_2 (v_2' - v_2)$$

I) $$m_1 (v_1 - v_1')(v_1 + v_1') = m_2 (v_2' - v_2)(v_2' + v_2)$$

$$\therefore \quad v_1 + v_1' = v_2' + v_2$$

III) $$\boxed{v_1 - v_2 = v_2' - v_1'}$$

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② elastischer Stoss, $m_1 = m_2$

II) $$v_1 - v_1' = v_2' - v_2$$

$$v_1 + v_2 = v_1' + v_2'$$

III) $$v_1 - v_2 = v_2' - v_1'$$

$$\therefore \quad \boxed{\begin{aligned} v_1 &= v_2' \\ v_2 &= v_1' \end{aligned}}$$

③ vollständig inelastischer Stoss

$$\longrightarrow \quad v_1' = v_2' = v'$$

Energiesatz:

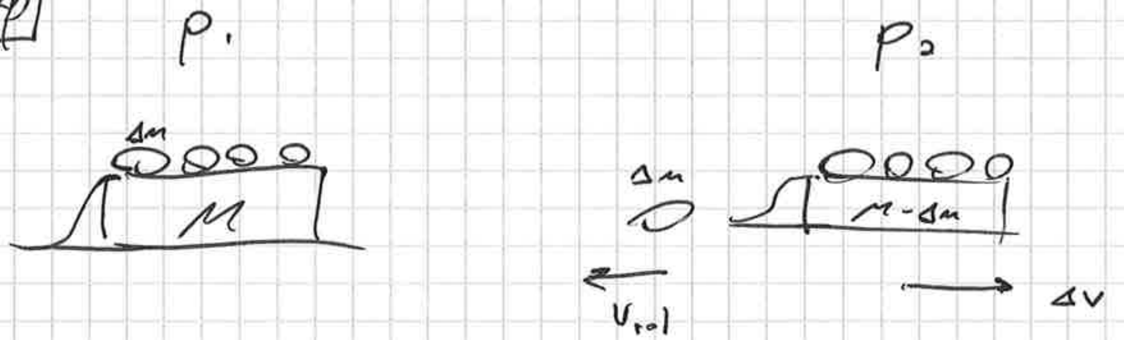
$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(m_1 + m_2) v'^2 + Q$$

Impulssatz:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v'$$

$$\boxed{v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}}$$

Exp



$$p_1 = p_2$$

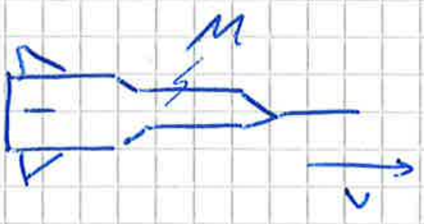
$$0 = -\Delta m \, v_{rel} + M \Delta v$$

$$\Delta v = \frac{\Delta m}{M} v_{rel}$$

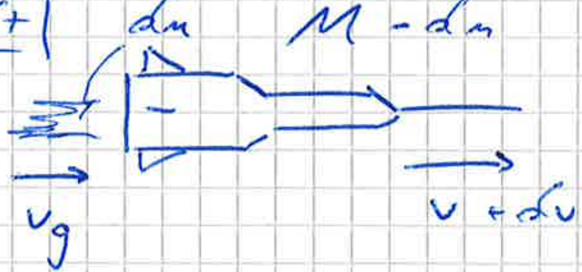
For fast rocket, fuel should be heavy & ejected w/ high velocity

Exp

$\boxed{\text{Zur Zeit } t}$



$\boxed{\text{Zur Zeit } t+dt}$



$$v_g = v + dv + v_{rel}$$

M : Masse der Rakete zur Zeit t

m : Masse der ausgestossenen Gase zur Zeit t

M_0 : Gesamtmasse

M_E : Masse der ausgebrannten Rakete

v : Geschwindigkeit der Rakete zur Zeit t

v_g : Geschwindigkeit der Gase zur Zeit t

v_{rel} : Relativgeschwindigkeit der ausgestossenen Gase

Impulssatz

$$p(t+dt) = p(t)$$

$$0 = p(t) - p(t+dt)$$

$$0 = Mv - \left\{ (M - dm)(v+dv) + dm(v + dv + v_{rel}) \right\}$$

$$0 = \cancel{Mv} - \cancel{Mv} - Mdv + \cancel{v\,dm} + \cancel{dm\,dv} - \cancel{v\,dm} - \cancel{dm\,dv} - v_{rel}\,dm$$

$$M dv = -v_{rel}\, dm \qquad \left(\begin{aligned} M &= M_0 - m \\ dM &= -dm \end{aligned} \right.$$

$$M\, dv = v_{rel}\, dM$$

$$dv = v_{rel} \frac{dM}{M}$$

②

$$\int_0^{V_E} dv = V_{rel} \int_{M_0}^{M_E} \frac{dM}{M} = -V_{rel} \int_{M_E}^{M_0} \frac{dM}{M}$$

$$\boxed{V_E = -V_{rel} \ln\left(\frac{M_0}{M_E}\right)}$$

→ Natürlicher Logarithmus

$\ln(x) = \log_e x$

Beispiel :

ausstossen

das Verhältnis $\frac{M_0}{M_E} = 50$, $v_{rel} = -1000\,m/s$

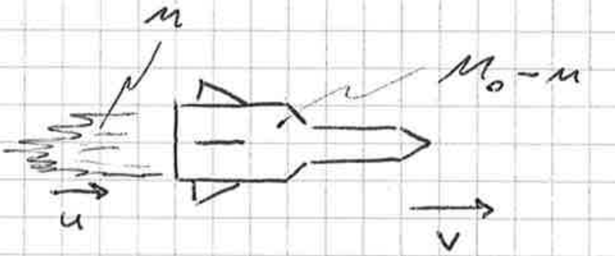
$\longrightarrow v_E = 1000\,m/s \;\ln(50)$

$v_E = 3912\,m/s \approx$ Mach 11

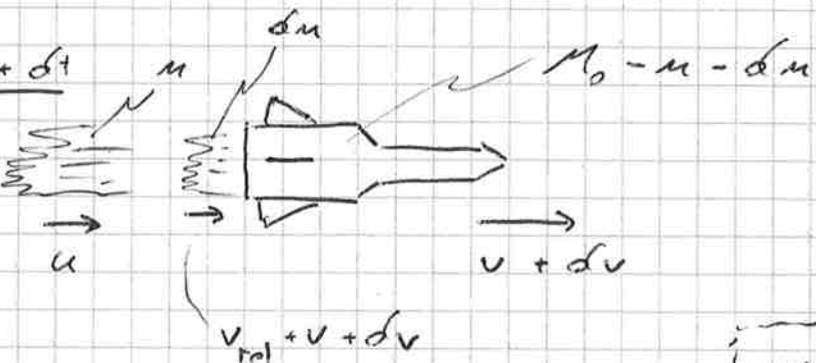
Schallgeschwindigkeit
343 m/s

Die Gase werden mit konstanter Geschwindigkeit v, 1000 m/s in Richtung −x ausgestossen.

Zeit t



Zeit $t + dt$



M_0 : Gesamtmasse
M_E : Masse der ausgebrannten Rakete
m : Masse des Treibstoffs zur Zeit t

v_{rel} : Relativgeschwindigkeit der ausgestossenen Gase zur Rakete

Impulssatz:

$$p(t+dt) = p(t)$$

$$0 = p(t+dt) - p(t) = dp$$

$$0 = (M_0 - m - dm)(v + dv) + mu + dm(v_{rel} + v + dv) - (M_0 - m)v - mu$$

$$0 = \cancel{M_0 v} + M_0 dv - \cancel{mv} - m dv - \cancel{v dm} - \cancel{dm dv} + \cancel{mu} + v_{rel} dm + \cancel{v dm} + \cancel{dm dv} - \cancel{M_0 v} + \cancel{mv} - \cancel{mu}$$

$$0 = M_0 dv - m dv + v_{rel} dm$$

$$\longrightarrow \quad dv = -v_{rel} \frac{1}{M_0 - m} dm$$

$$\int_0^{v_E} dv = -v_{rel} \int_0^{M_0 - M_E} \frac{dm}{M_0 - m}$$

M : Masse der Rakete zur Zeit t
$M = M_0 - m$
$dM = -dm$

$$v_E = +v_{rel} \int_{M_0}^{M_E} \frac{dM}{M} = -v_{rel} \int_{M_E}^{M_0} \frac{dM}{M}$$

$$\longrightarrow \quad \boxed{v_E = -v_{rel} \ln\left(\frac{M_0}{M_E}\right)}$$

n.b.: $v_{rel} < 0$

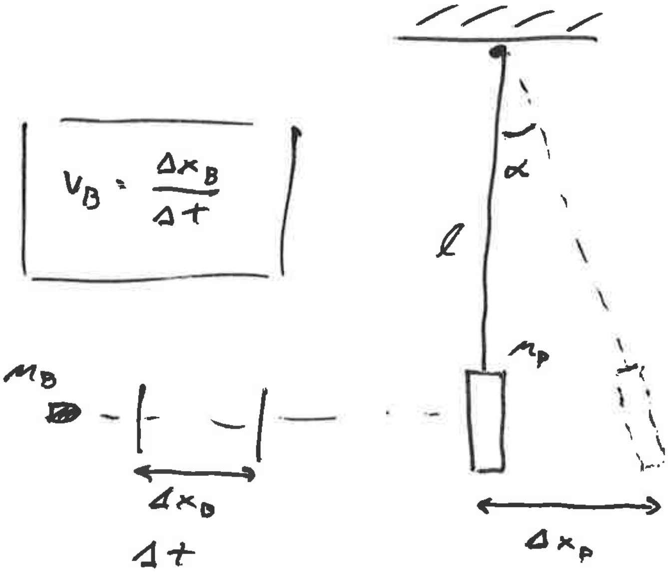
$\boxed{\text{Exp}}$ Inelastic collision with Pendulum

- Experiment
- Energy not conserved ($Q > 0$): energy lost in deformation of block and bullet and in heat.
- Momentum conserved ($F_{ext} = 0$).

We know: mass of bullet m_B
mass of pendulum m_p
length of pendulum ℓ

We measure: speed of bullet v_B
displacement of pendulum Δx_p

$$\boxed{v_B = \frac{\Delta x_B}{\Delta t}}$$



From before:

$$v_p = \sqrt{2 g \ell (1 - \cos\alpha)}$$

For small α,

$$\alpha \simeq \frac{\Delta x_p}{\ell} \quad \text{and}$$

$$(1 - \cos\alpha) \simeq \frac{1}{2} \frac{\Delta x_p^2}{\ell^2}$$

$$\therefore \quad V_P = \sqrt{\cancel{2} g\, \cancel{l} \cdot \frac{1}{\cancel{2}} \frac{\Delta x_P^2}{l^{\cancel{2}}}}$$

$$V_P = \sqrt{\frac{g}{l}}\, \Delta x_P$$

Conservation of Momentum

$$m_B V_B = (m_B + m_P) V_P$$

$$V_B = \left(\frac{m_B + m_P}{m_B} \right) V_P$$

$$\boxed{V_B = \left(\frac{m_B + m_P}{m_B} \right) \sqrt{\frac{g}{l}}\, \Delta x_P}$$

Drehimpulssatz

$$\vec{L}_{tot} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\frac{d\vec{L}_{tot}}{dt} = \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) = \sum_i \frac{d\vec{r}}{dt} \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

$$= \underbrace{\sum_i \vec{v}_i \times m\vec{v}_i}_{0} + \sum_i \vec{r}_i \times \vec{F}_i$$

weil $\vec{v}_i \times \vec{v}_i = 0$

$$\frac{d\vec{L}_{tot}}{dt} = \sum_i \vec{M}_i = \vec{M}_{tot}$$

$\therefore$ Wenn $\sum_i \vec{M}_i = 0$, $\vec{L}_{tot}$ = Konstant.

$$\vec{L} = J_0 \vec{\omega}_0 = J_1 \vec{\omega}_1$$

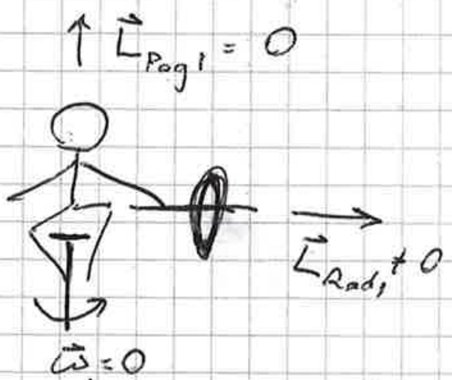
$\boxed{\text{Exp}}$

$$J_1 < J_0$$

$$\hookrightarrow \vec{\omega}_1 > \vec{\omega}_0$$

$\boxed{\text{Exp}}$

①



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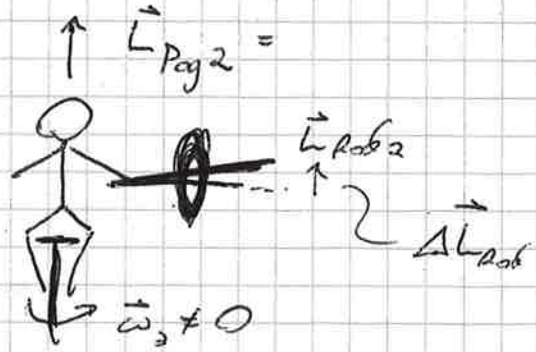

$$\vec{L}_{Pog_1} + \vec{L}_{Rad_1} = \vec{L}_{Pog2} + \vec{L}_{Rad_2} \qquad \vec{L}_{Pog2} = -\Delta\vec{L}_{Rad}$$

$\boxed{\text{Exp}}$ Gyro

$$0 + \vec{L}_{Rad1} = \vec{L}_{Pog2} + \cancel{\vec{L}_{Rad1}} + \Delta\vec{L}_{Rad}$$

gyroscope

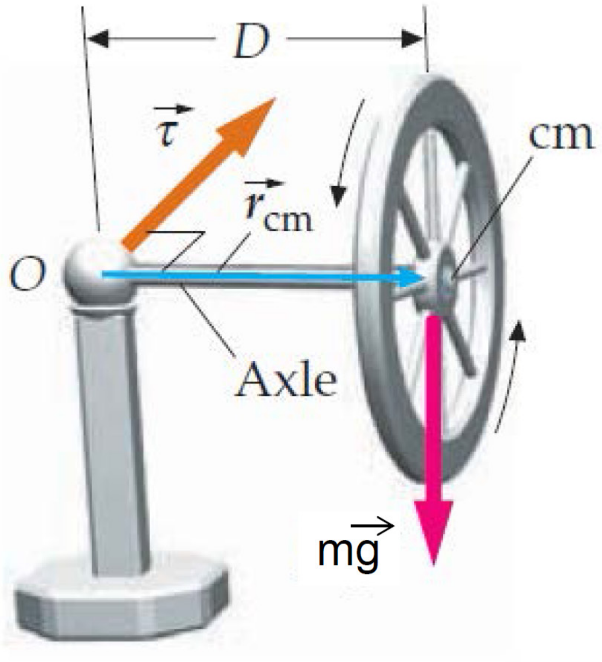

FIGURE 10-22

Tipler, Mosca, Physics, 6th ed. (2008)

gyroscope

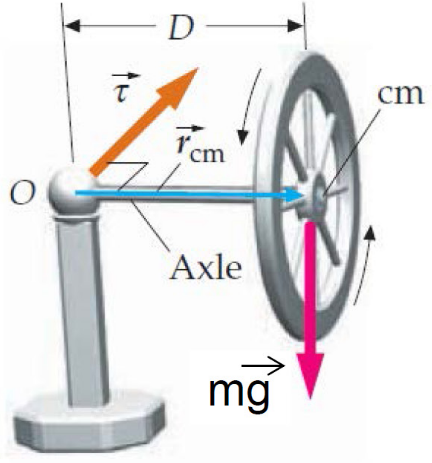

FIGURE 10-22

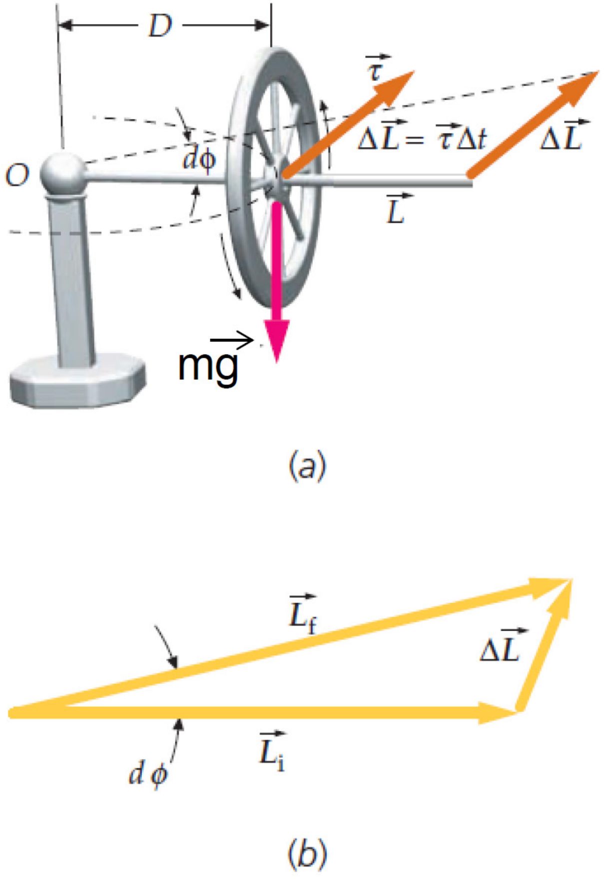

FIGURE 10-23

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