Concepts in Mesoscopic Physics

Drude Conductivity

$$\sigma = en\mu = \frac{ne^2\tau_m}{m^*}$$

rewrite using $k_F = \sqrt{2\pi n}$

$$\ell = v_F \tau_m$$

$$v_F = \frac{\hbar k_F}{m^*}$$

$$\boxed{\sigma = g_s g_v \frac{e^2}{h}\frac{k_F \ell}{2} = \frac{2e^2}{h}\frac{k_F \ell}{2}}$$

rewrite using

$$\rho_{DOS} = \frac{g_s g_v m^*}{2\pi\hbar^2} = \frac{m^*}{\pi\hbar^2}$$

$$D = \frac{1}{2}v_F^2\tau_m = \frac{1}{2}v_F\ell$$

$$\boxed{\sigma = e^2\rho_{DOS}(E)D}$$

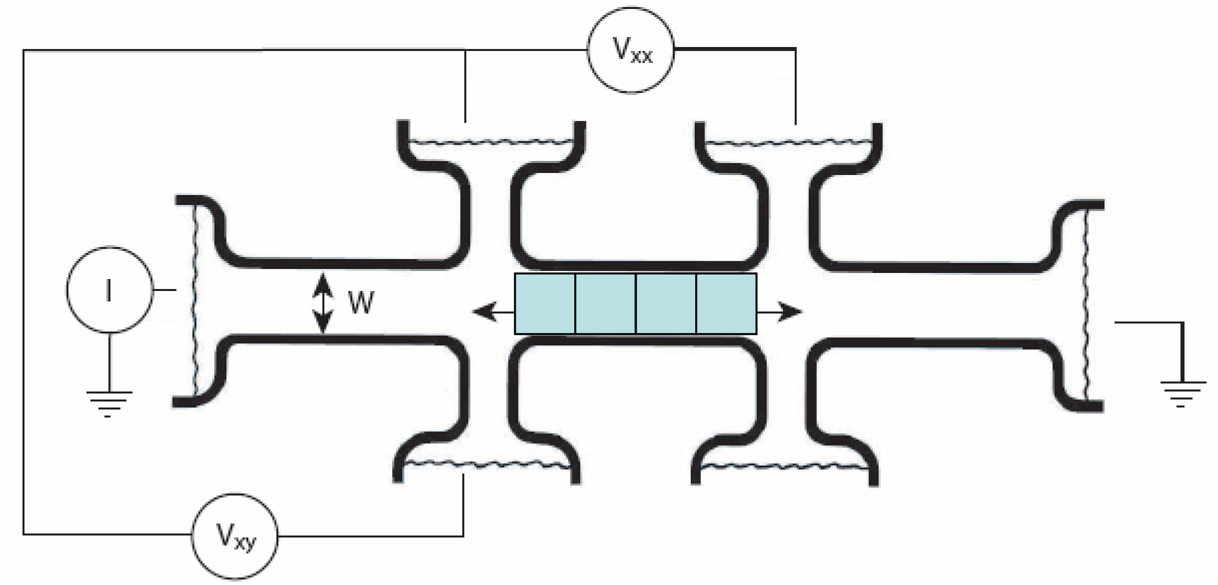
Resistance per Square

2D: resistivity and resistance: same units

$$R = \rho \frac{L}{W} = \rho_{\square} \frac{L}{W}$$

$\rho_{\square}$ resistance per square

example: Hall bar



Mesoscopic Time and Length Scales

Femi wavelength λ_F

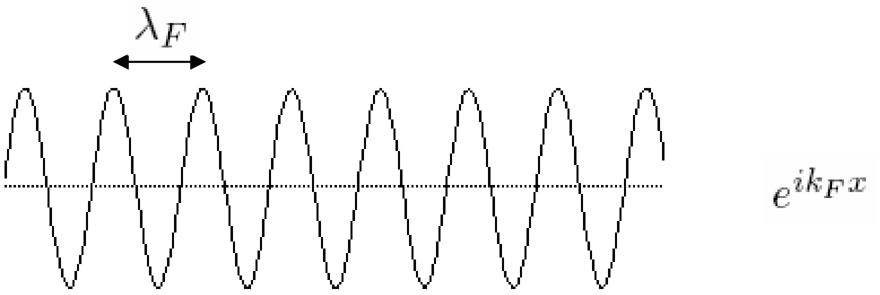
$$\lambda_F = 2\pi/k_F = \sqrt{2\pi/n}$$

Typically, $n \sim 2\times10^{11}\,\mathrm{cm}^{-2} = 2\times10^{15}\,\mathrm{m}^{-2}$

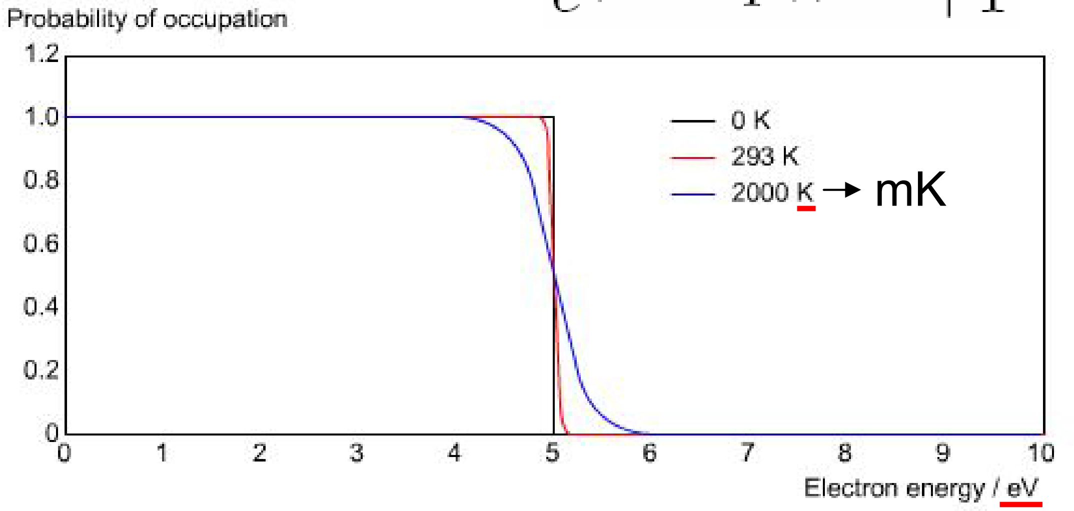
$$E_F \sim 7\,\mathrm{meV}$$

$$m^* = 0.067 m_e$$

$$\lambda_F \sim 56\,\mathrm{nm}$$

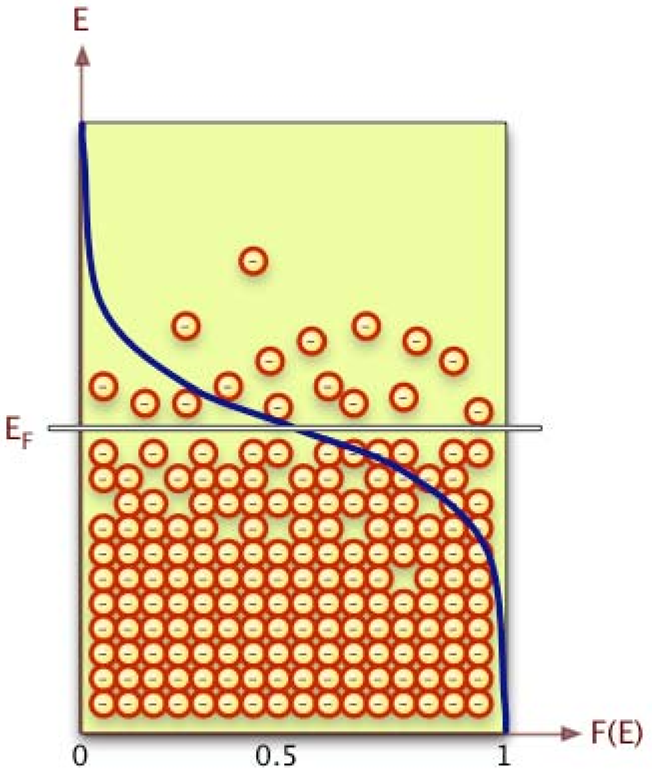


Fermi-Dirac distribution $\dfrac{1}{e^{(E-E_F)/kT}+1}$



Fermi-Dirac distribution for several temperatures

meV



Mesoscopic Time and Length Scales

Mean free path ℓ

$$\ell = v_F \tau_m = v_F \mu \frac{m^*}{e}$$

$$\mu = 100\,\mathrm{m}^2/(\mathrm{Vs}) = 1'000'000\,\mathrm{cm}^2/(\mathrm{Vs})$$

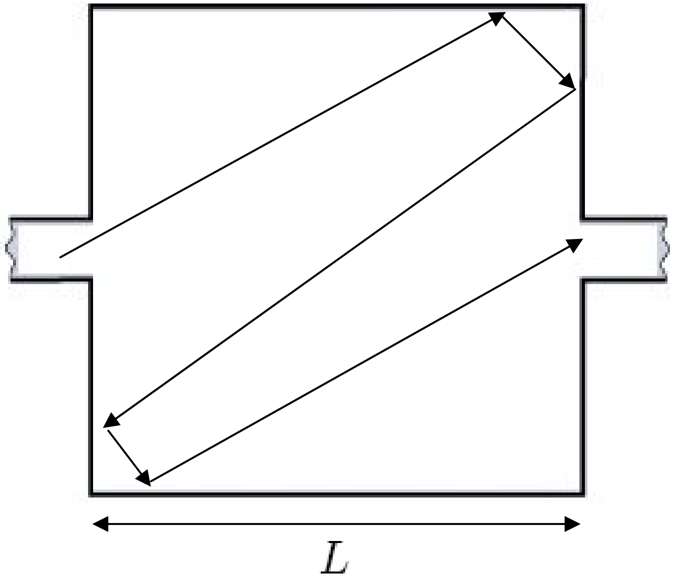
$$\tau_m = 38\,\mathrm{ps}$$

$$\ell = 7.4\,\mu\mathrm{m}$$

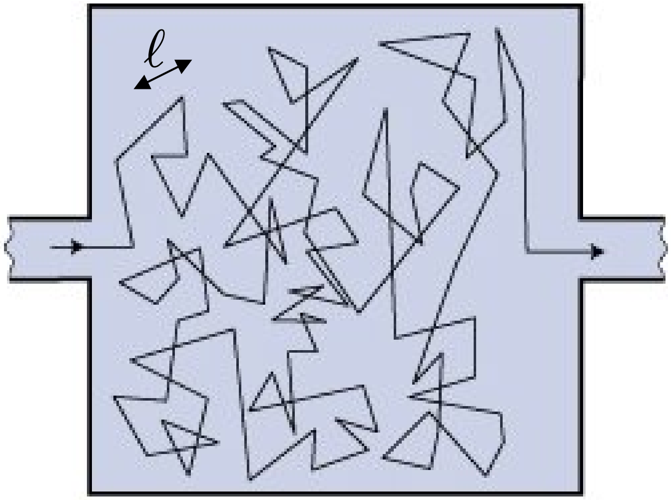
diffusion constant $D = \frac{1}{2} v_F^2 \tau_m$

$$D = 0.72\,\mathrm{m}^2/\mathrm{s}$$

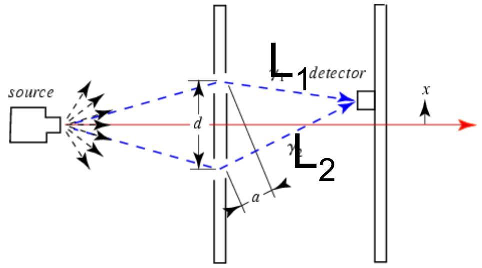
ballistic $\ell \gg L$



diffusive $\ell \ll L$



Mesoscopic Time and Length Scales



Phase coherence time τ_φ

interference $|A_1 + A_2|^2 \sim Re \exp\left(ik_F(L_1 - L_2) + i(\varphi_1 - \varphi_2)\right)$

phase coherence length L_φ

interference suppressed to 1 / e

$\exp(-L/L_\varphi)$

phase coherence time τ_φ

$L_\varphi = v_F \tau_\varphi$ ballistic

$L_\varphi = \sqrt{D\tau_\varphi}$ diffusive

due to interactions... $\varphi(t)$ randomized

$$\langle\varphi\rangle_t = \int_0^t \varphi(\tau)d\tau \sim 0$$

$$\langle\exp(i\varphi(\tau))\rangle_t \sim \exp(-t/\tau_\varphi)$$

quasi one-dimensional $L \ll L_\varphi$

Mesoscopic Time and Length Scales

Phase coherence time τ_φ

finite due to coupling of electrons to environment:
dynamic scattering mechanisms

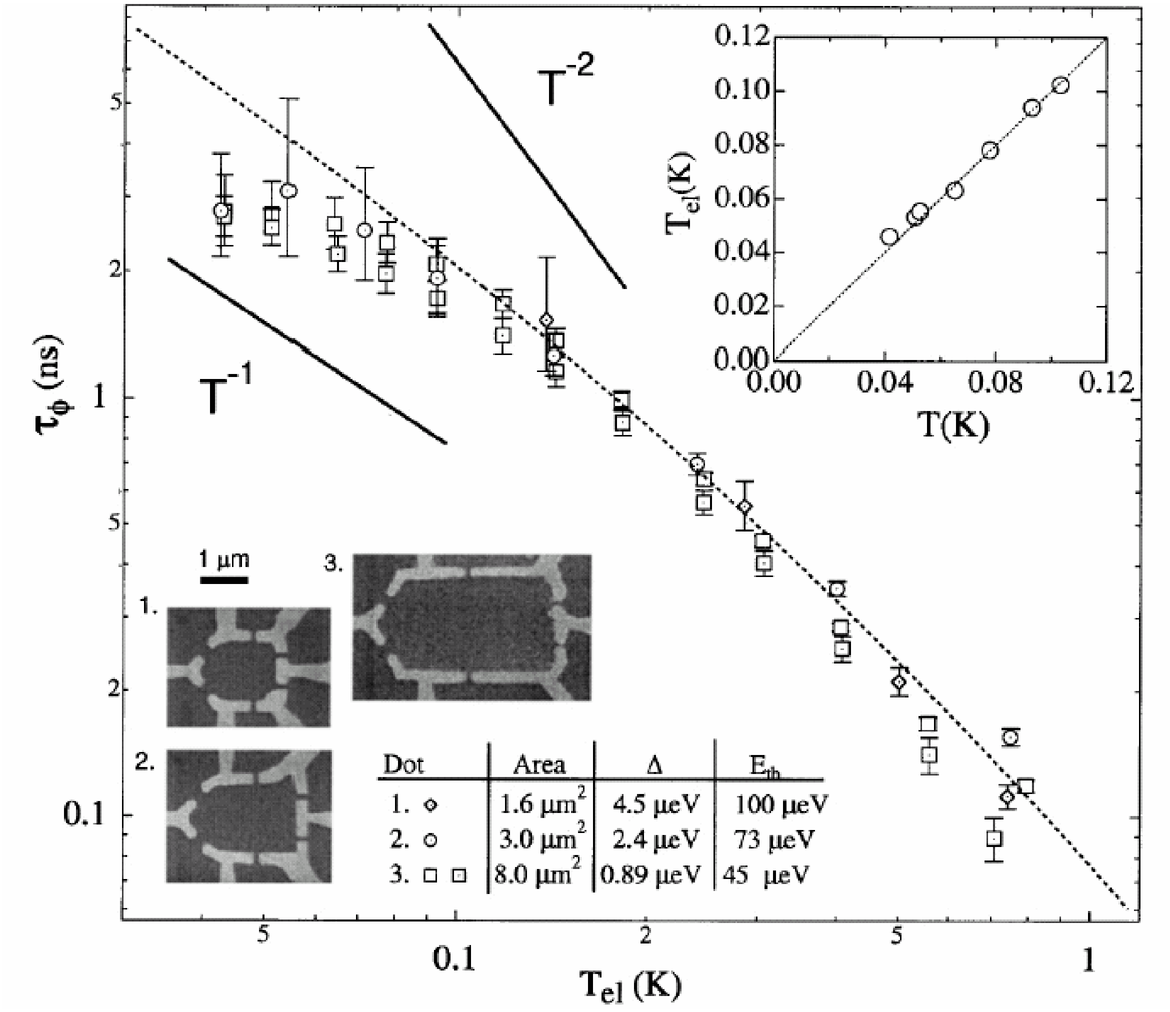
- electron phonon scattering
- electron – electron scattering
  a) large energy exchange
  b) quasi elastic scattering (Nyquist mechanism)
- spin flip scattering (magnetic impurities)
- electron – photon scattering
- etc

$$\tau_\varphi = \tau_\varphi(T)$$

Mesoscopic Time and Length Scales

Phase coherence time τ_φ

Huibers et al., PRL 1999



Dot	Area	Δ	E_{th}
1. ◇	1.6 µm^2	4.5 µeV	100 µeV
2. ○	3.0 µm^2	2.4 µeV	73 µeV
3. □ □	8.0 µm^2	0.89 µeV	45 µeV

$$\tau_\varphi[\mathrm{ns}] = (4.0T[K] + 9.0T[K]^2)^{-1}$$

e-e direct

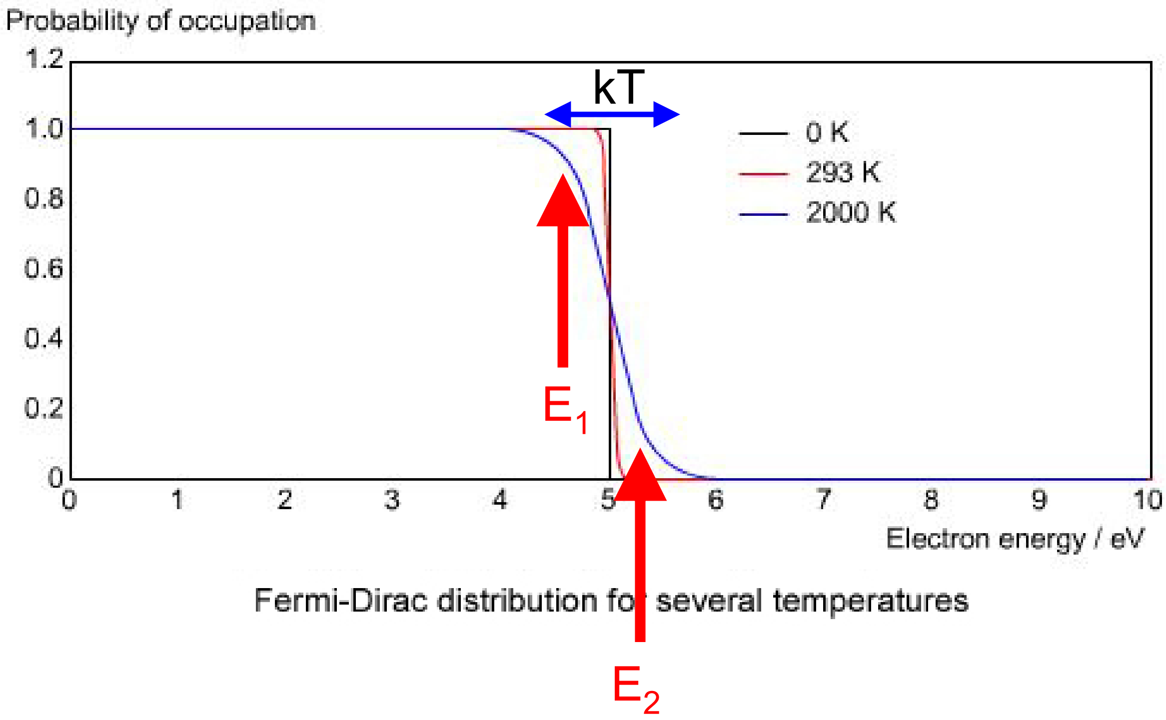
$$\tau_{ee}^{-1} = \frac{\pi}{4}\frac{(k_B T)^2}{\hbar E_F}\ln\frac{E_F}{k_B T}$$

e-e quasi-elastic (Nyquist)

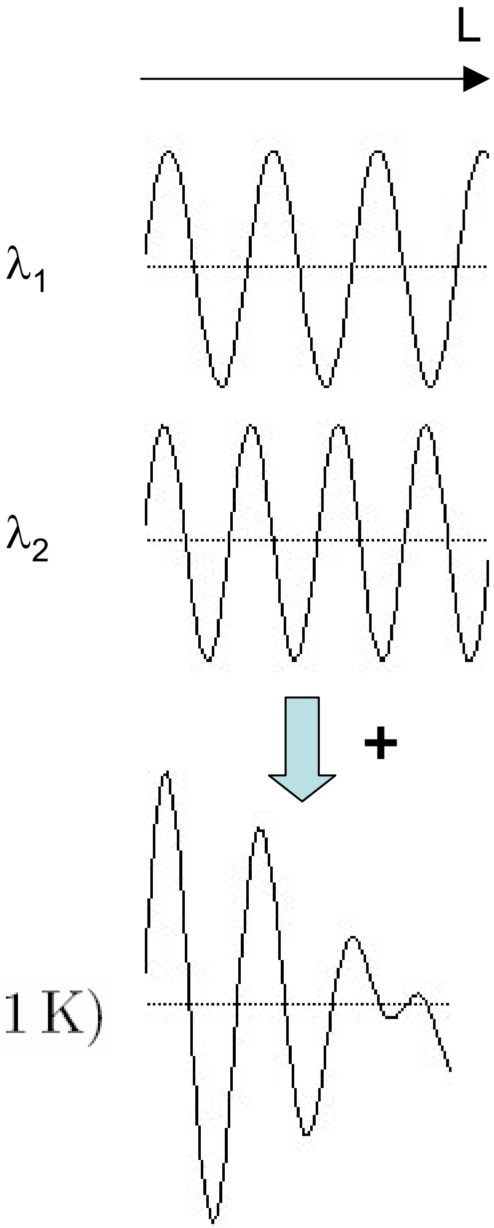
$$\tau_{\phi N}^{-1} = \frac{k_B T}{2\pi\hbar}\frac{\lambda_F}{\ell_e}\ln\frac{\pi\ell_e}{\lambda_F}$$

e-phonon: small (T < 1K)

Thermal length L_T : thermal smearing



Fermi-Dirac distribution for several temperatures



$$(k(E_F + kT) - k(E_F))x = 1$$

ballistic $L_T = \frac{\hbar v_F}{kT}$ $L_T \sim 7.4\,\mu\mathrm{m}\ (\text{for } T = 0.1\,\mathrm{K})$

diffusive $L_T = \sqrt{\frac{\hbar D}{kT}}$

Interaction parameter r_S

ratio of Coulomb energy to kinetic energy

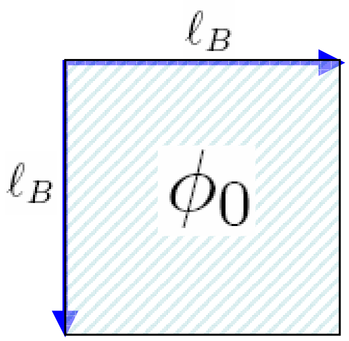
$$r_S = \frac{e^2}{4\pi\epsilon_0\epsilon r} \div E_F = \frac{e^2 m^*}{\epsilon\epsilon_0 h^2}\frac{1}{\sqrt{n}} \sim 0.7$$

characterizes “strength” of electron interactions

non interacting
weakly interacting $r_s \to 0$

strongly interacting $r_s \gtrsim 1$

Magnetic Length ℓ_B



$\otimes$ B

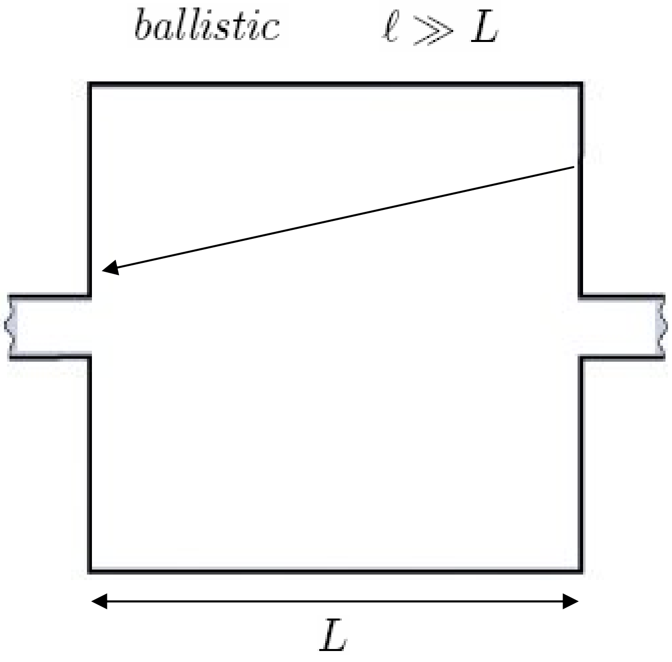
$$\phi_0 = \frac{h}{e}$$

flux quantum

magnetic flux = A · B = L^2 · B

$$\ell_B = \sqrt{\frac{\hbar}{eB}}$$

Thouless Energy



crossing time: L/v_F

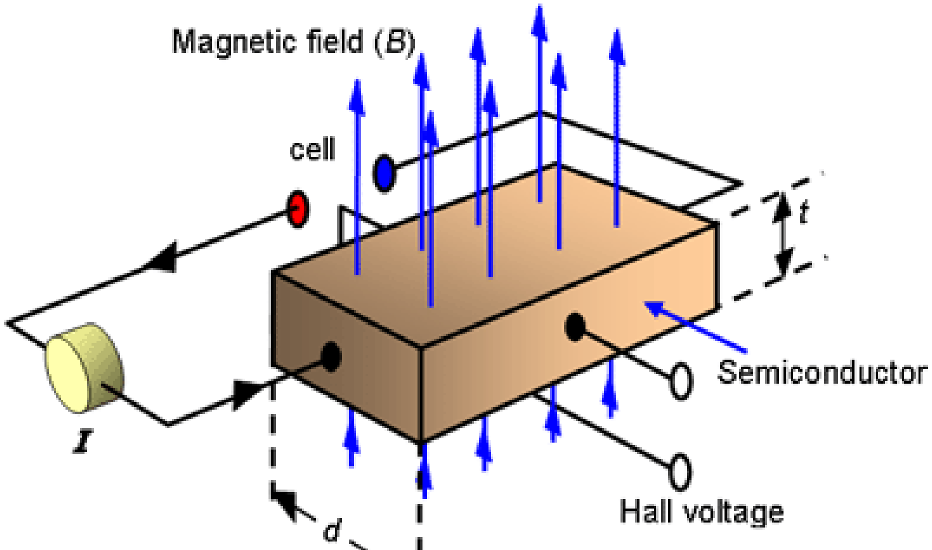
Thouless energy: $E_T = \frac{\hbar V_F}{L}$

$$E_T = \frac{\hbar V_F}{\sqrt{A}}$$

TABLE I Electronic properties of the 2DEG in GaAs-AlGaAs heterostructures and Si inversion layers.

		GaAs(100)	Si(100)	Units
Effective Mass	m	0.067	0.19	$m_{\mathrm{e}} = 9.1 \times 10^{-28}$ g
Spin Degeneracy	g_{s}	2	2	
Valley Degeneracy	g_{v}	1	2	
Dielectric Constant	ε	13.1	11.9	$\varepsilon_0 = 8.9 \times 10^{-12}$ Fm^{-1}
Density of States	$\rho(E) = g_{\mathrm{s}} g_{\mathrm{v}} (m/2\pi\hbar^2)$	0.28	1.59	10^{11} cm^{-2} meV^{-1}
Electronic Sheet Density[a]	n_{s}	4	1–10	10^{11} cm^{-2}
Fermi Wave Vector	$k_{\mathrm{F}} = (4\pi n_{\mathrm{s}}/g_{\mathrm{s}} g_{\mathrm{v}})^{1/2}$	1.58	0.56–1.77	10^6 cm^{-1}
Fermi Velocity	$v_{\mathrm{F}} = \hbar k_{\mathrm{F}}/m$	2.7	0.34–1.1	10^7 cm/s
Fermi Energy	$E_{\mathrm{F}} = (\hbar k_{\mathrm{F}})^2/2m$	14	0.63–6.3	meV
Electron Mobility[a]	μ_{e}	$10^4 - 10^6$	10^4	cm^2/Vs
Scattering Time	$\tau = m\mu_{\mathrm{e}}/e$	0.38–38	1.1	ps
Diffusion Constant	$D = v_{\mathrm{F}}^2 \tau/2$	140–14000	6.4–64	cm^2/s
Resistivity	$\rho = (n_{\mathrm{s}} e \mu_{\mathrm{e}})^{-1}$	1.6–0.016	6.3–0.63	kΩ
Fermi Wavelength	$\lambda_{\mathrm{F}} = 2\pi/k_{\mathrm{F}}$	40	112–35	nm
Mean Free Path	$l = v_{\mathrm{F}} \tau$	$10^2 - 10^4$	37–118	nm
Phase Coherence Length[b]	$l_\phi = (D\tau_\phi)^{1/2}$	200–...	40–400	nm$(T/\mathrm{K})^{-1/2}$
Thermal Length	$l_{\mathrm{T}} = (\hbar D/k_{\mathrm{B}} T)^{1/2}$	330–3300	70–220	nm$(T/\mathrm{K})^{-1/2}$
Cyclotron Radius	$l_{\mathrm{cycl}} = \hbar k_{\mathrm{F}}/eB$	100	37–116	nm$(B/\mathrm{T})^{-1}$
Magnetic Length	$l_{\mathrm{m}} = (\hbar/eB)^{1/2}$	26	26	nm$(B/\mathrm{T})^{-1/2}$
	$k_{\mathrm{F}} l$	15.8–1580	2.1–21	
	$\omega_{\mathrm{c}} \tau$	1–100	1	(B/T)
	$E_{\mathrm{F}}/\hbar\omega_{\mathrm{c}}$	7.9	1–10	$(B/\mathrm{T})^{-1}$

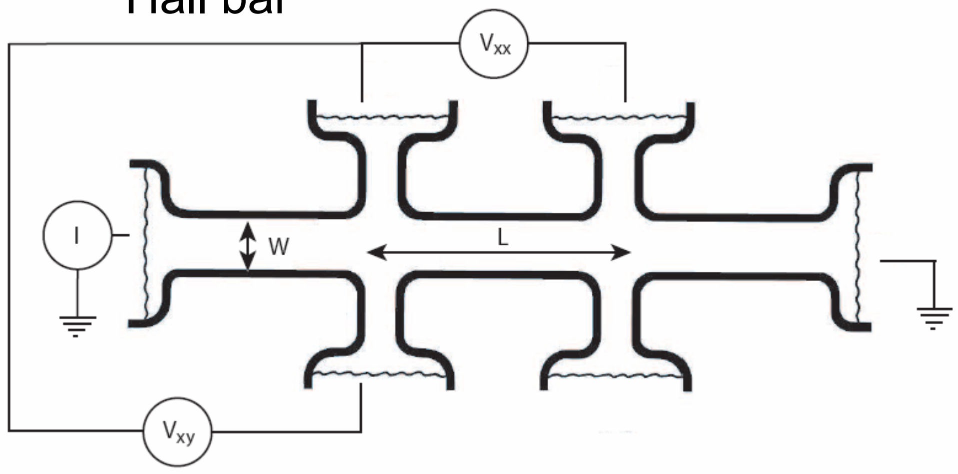
Classical Hall Effect



(resourcefulphysics.org)

2D: thickness t drops out

Hall bar

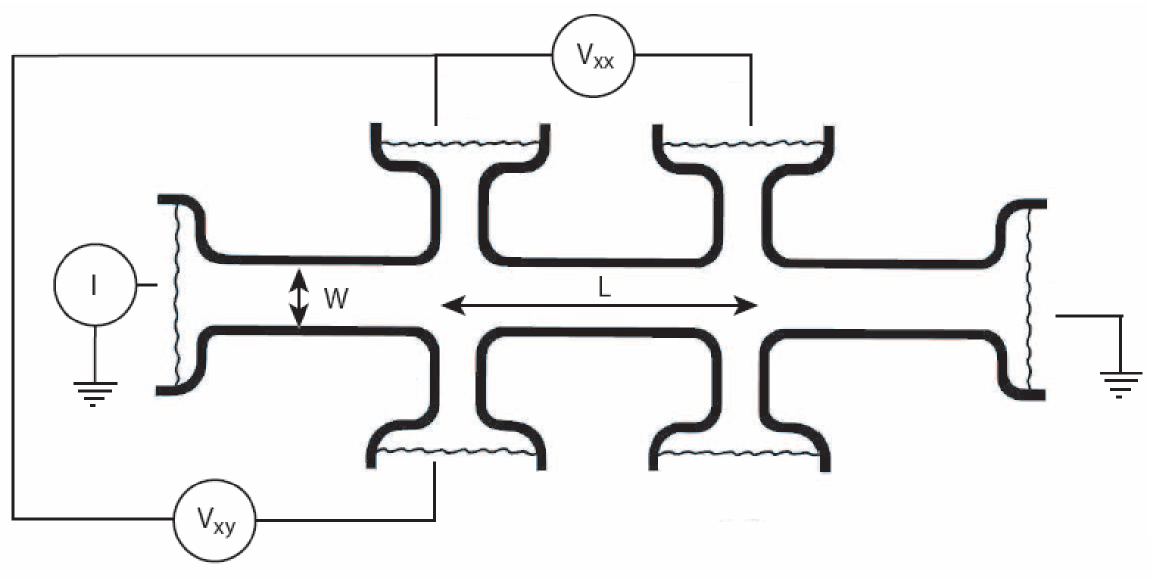


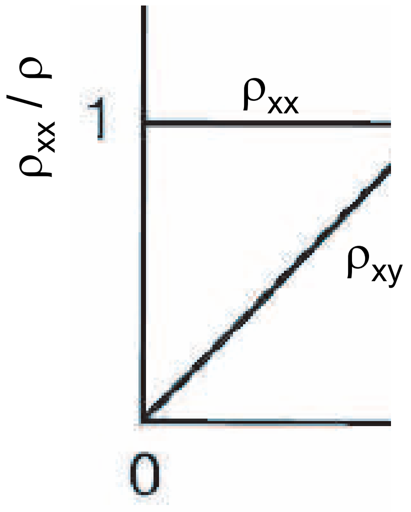
$$\frac{m^* v_d}{\tau_m} = e\,[E + v_d \wedge B]$$

$$\begin{pmatrix} \frac{m^*}{e\tau_m} & -B \\ +B & \frac{m^*}{e\tau_m} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\rho_{xx} = \sigma^{-1}, \quad \rho_{xy} = -\rho_{yx} = -\frac{B}{en}$$

Classical Hall Effect





$$V_x = R_{xx} I_x \qquad R_{xx} = \frac{L}{W}\rho_{xx}$$

$$V_H = V_y = \rho_{yx} I_x = \frac{B}{en} I_x = R_H I_x$$

$$R_H = B/(en)$$

$$R_H \sim 3.1\,\mathrm{k\Omega/Tesla}\ \text{for}\ n \sim 2 \times 10^{-11}\,\mathrm{cm}^{-2}$$

$$\mu = (neR_{xx}W/L)^{-1}$$

Quantum Hall Effect

Phenomenological Treatment

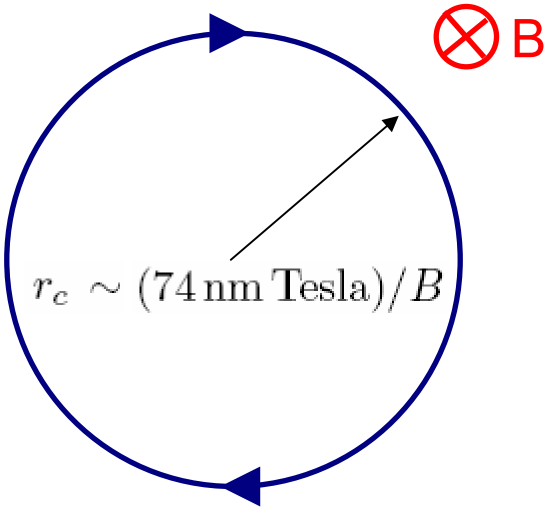
Lorentz-Force $F = ev \times B$

zentrifugal force $m\omega^2 r$

$$evB = m\omega^2 r$$

$$\omega_c = \frac{eB}{m^*}, \quad r_c = \frac{v}{\omega_c}$$

cyclotron frequency, radius

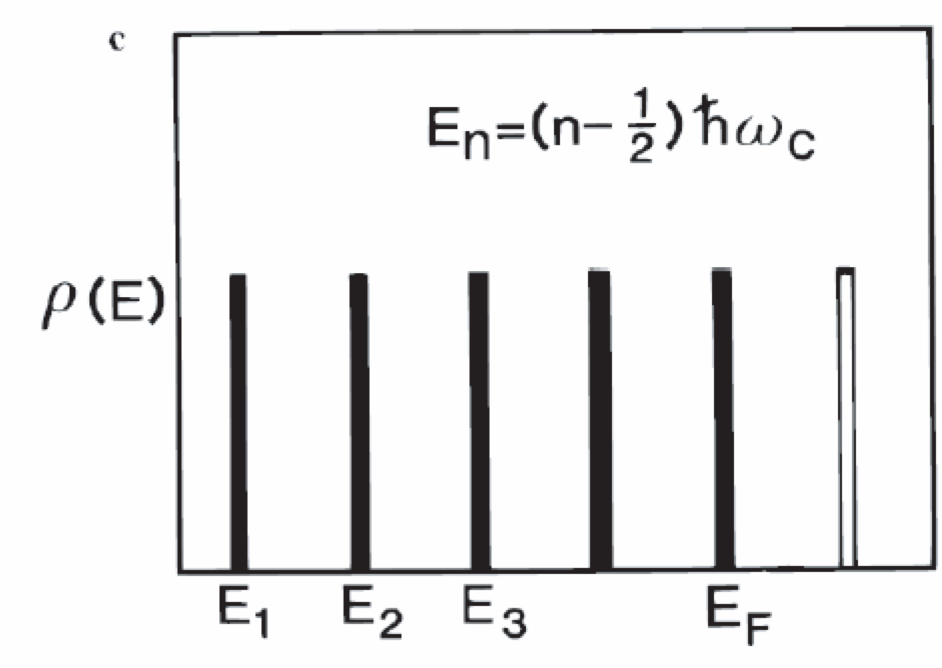


$$\hbar\omega_c = (1.73\,\mathrm{meV}/\mathrm{Tesla})$$
$$1\,\mathrm{meV}/k_B = 11.6K$$

quantization condition: circumference = N λ_F ⇨ $E_n = n\frac{1}{2}\hbar\omega_c$

Quantum Hall Effect

density of states



Landau levels

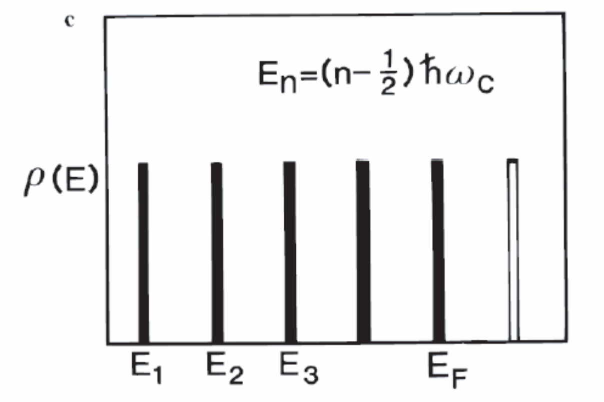
$$\rho_{DOS}(E,B) = N_0 \sum_{n=0}^{\infty} \delta\left(E-(n+1/2)\hbar\omega_c\right)$$

N_0 number of states per area in each Landau level

Quantum Hall Effect

all zero field states within range in energy of $\hbar\omega_c$ condense in one Landau level

$$N_0 = \hbar\omega_c \times (m/(\pi\hbar^2)) = 2eB/h.$$



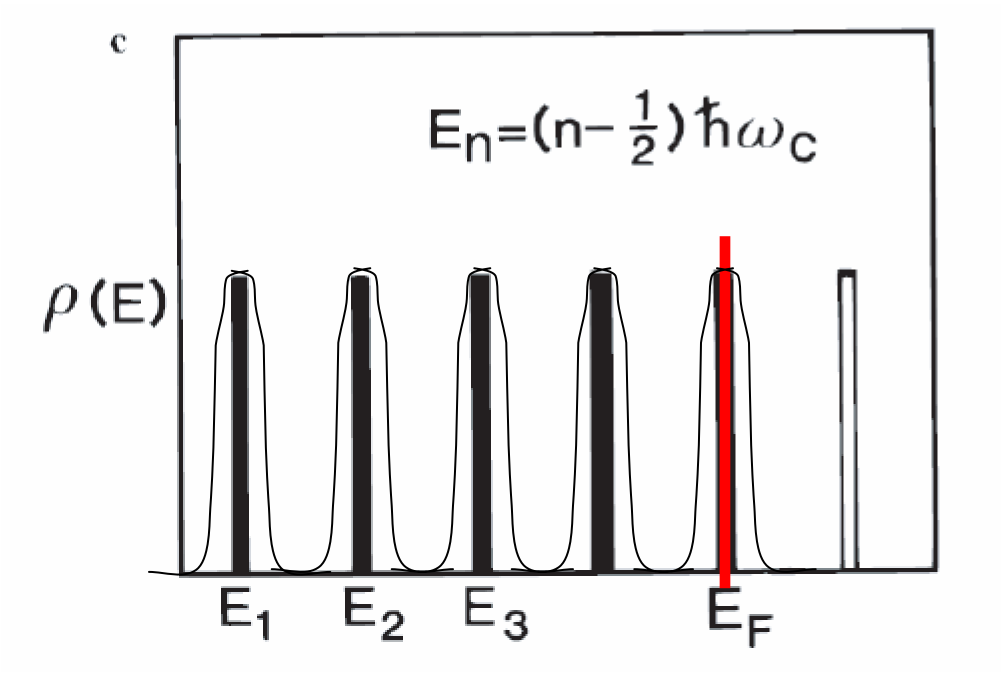
Landau level filling factor

$$\nu = \frac{n}{2eB/h}$$

number N of Landau levels with $E < E_F$: integer

Quantum Hall Effect: Transport

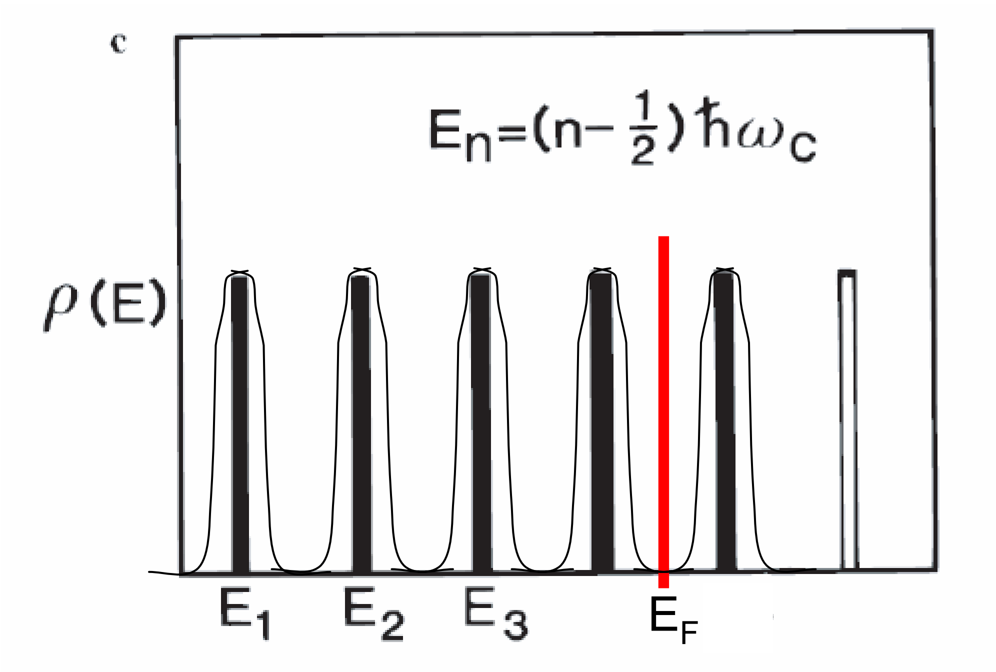
$\nu = N$: scattering possible, resistance



disorder broadening

Quantum Hall Effect: Transport

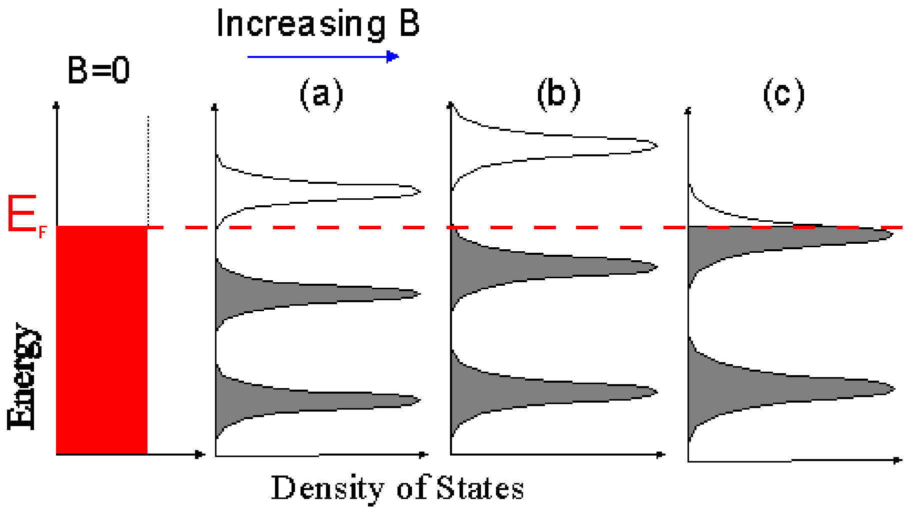
$\nu <> N$: scattering NOT possible, ZERO resistance (ρ_{xx})

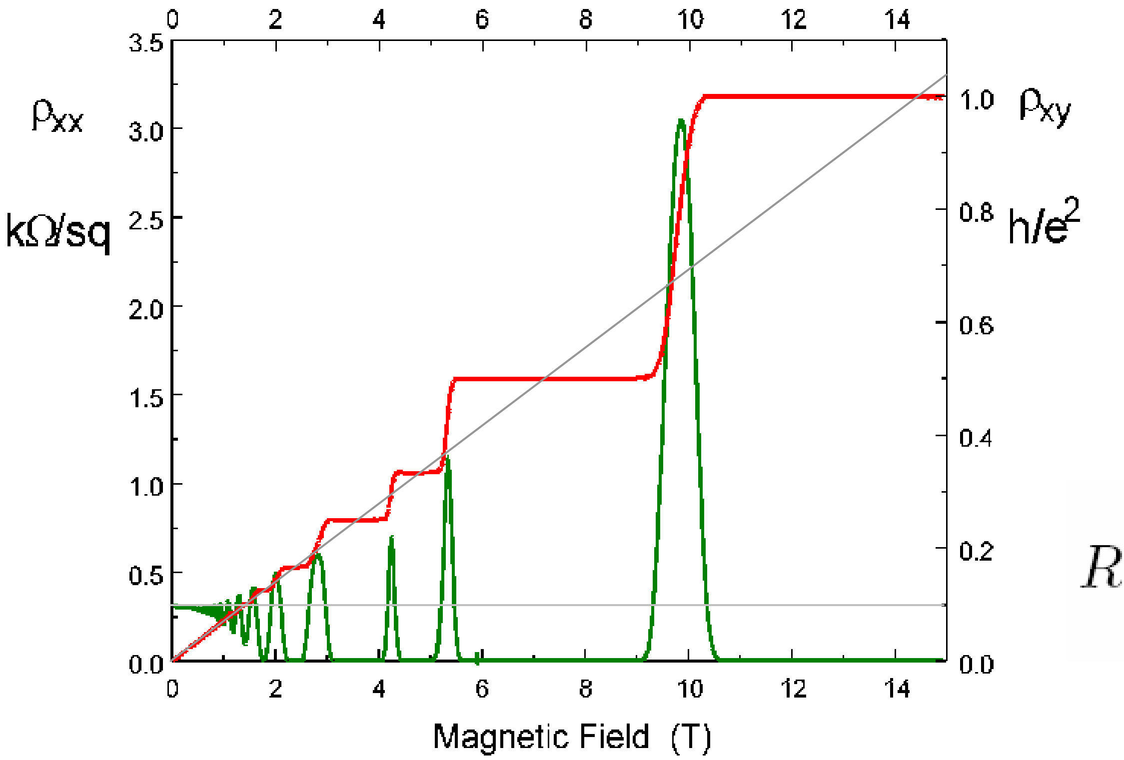


$$\hbar\omega_c \gg kT$$

disorder broadening

Quantum Hall Effect: B dependence

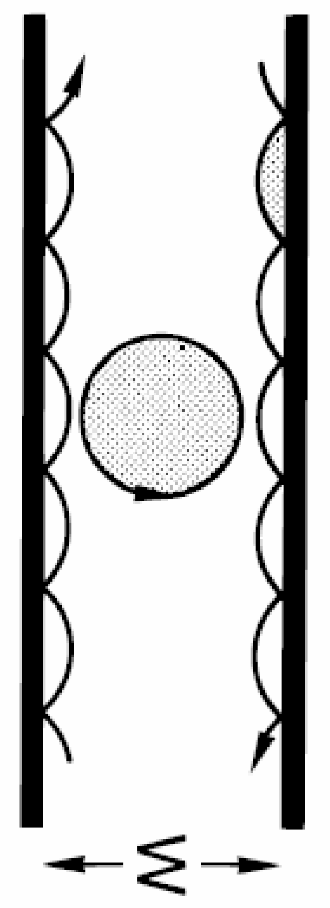


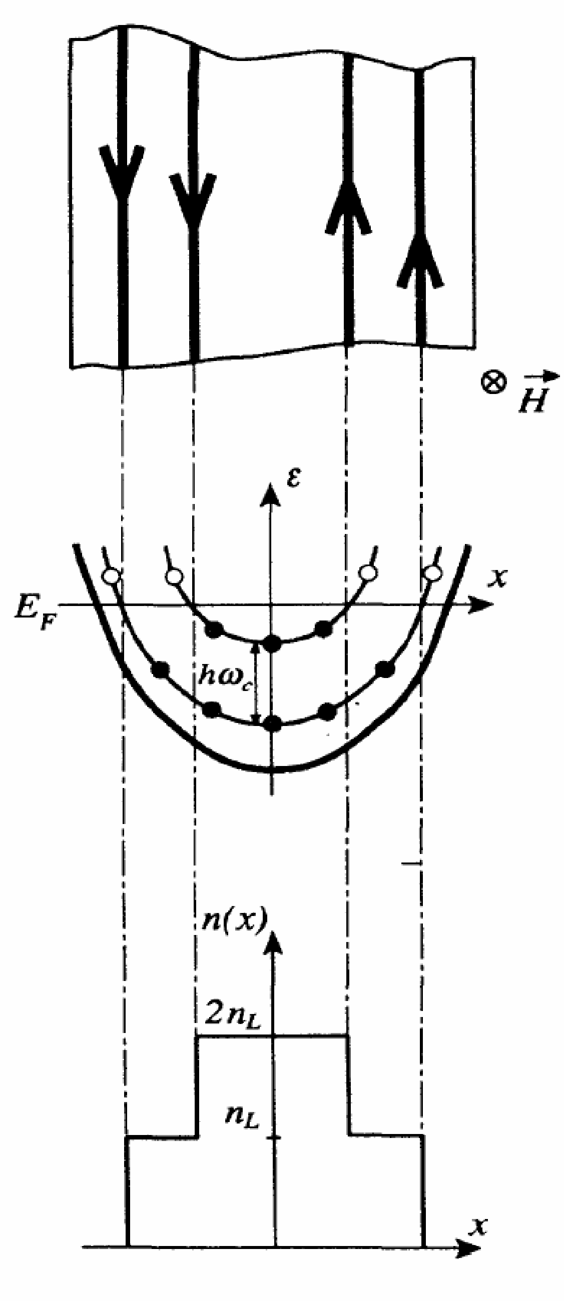


Integer Quantum Hall Effect

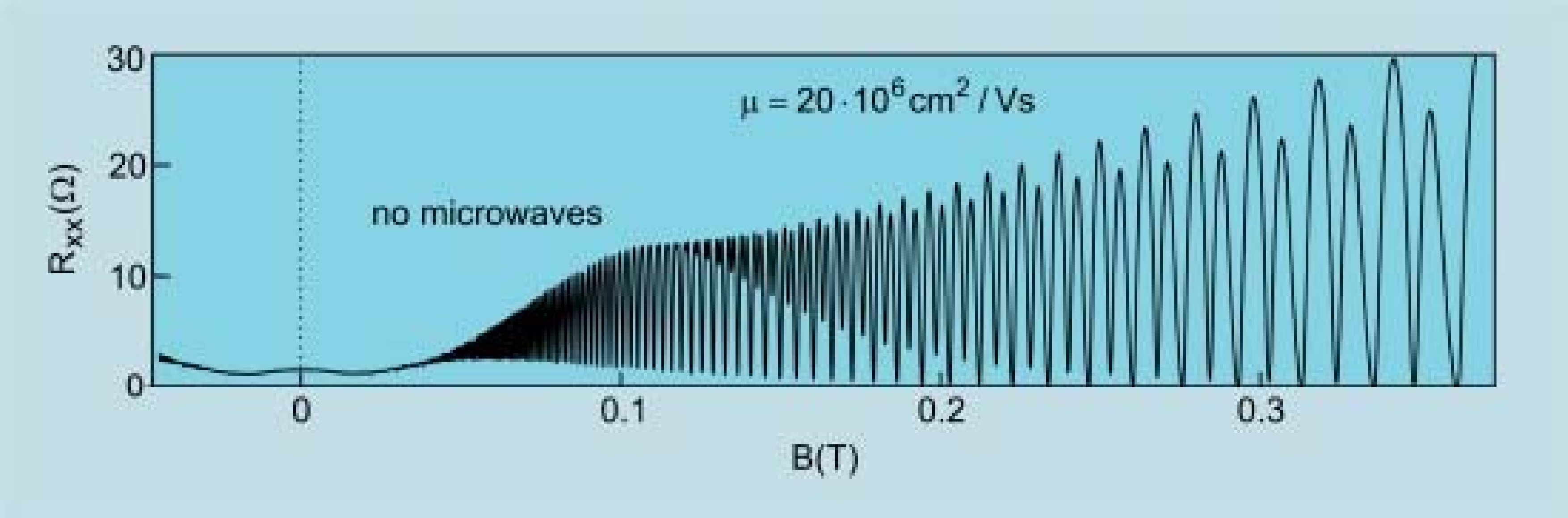
$$R_H = \frac{1}{g_s g_v}\frac{h}{e^2}\frac{1}{N} = \frac{h}{2e^2}\frac{1}{N}$$

Quantum Hall Effect: edge states / skipping orbits





Shubnikov-de-Haas Oscillations



$\hbar\omega_c > kT, \hbar/\tau_m$ orbital Landau levels

$g\mu_B B > kT, \hbar/\tau_m$ spin polarized Landau levels

Quantum Mechanical Treatment

$$\left[\frac{(i\hbar\nabla + eA)^2}{2m^*} + U(y)\right]\psi(x,y) = E\psi(x,y)$$

$$A = -\hat{x}By \quad \rightarrow \quad A_x = -By \quad \text{and} \quad A_y = 0.$$

three cases:
- free electrons in magnetic field (U = 0)
- confined, B = 0 (constriction, QPC)
- confined *and* in magnetic field

$$\left[\frac{(i\hbar\nabla + eA)^2}{2m^*} + U(y)\right]\psi(x,y) = E\psi(x,y)$$

$$p_x = -i\hbar\frac{\partial}{\partial x} \quad \text{and} \quad p_y = i\hbar\frac{\partial}{\partial y}$$

$$\left[\frac{(p_x + eBy)^2}{2m^*} + \frac{p_y^2}{2m^*} + U(y)\right]\psi(x,y) = E\psi(x,y)$$

$$\psi(x,y) = \frac{1}{\sqrt{L}}\exp(ikx)\chi(y)$$

$$\left[\frac{(\hbar k + eBy)^2}{2m^*} + \frac{p_y^2}{2m^*} + U(y)\right]\chi(y) = E\chi(y)$$

Free electrons in a magnetic field

$$U \equiv 0$$

$$\left[\frac{p_y^2}{2m^*} + \frac{1}{2} m^* \omega_c^2 (y + y_k)^2\right] \chi(y) = E\chi(y)$$

harmonic oscillator

$$y_k = \frac{\hbar k}{eB} \quad \text{and} \quad \omega_c = \frac{eB}{m^*}$$

... textbook ...

$$E(n,k) \;=\; \left(n + \underline{\frac{1}{2}}\right) \hbar\omega_c, \quad n = 0, 1, 2, \ldots .$$

zero point energy
pure quantum effect
no classical analog

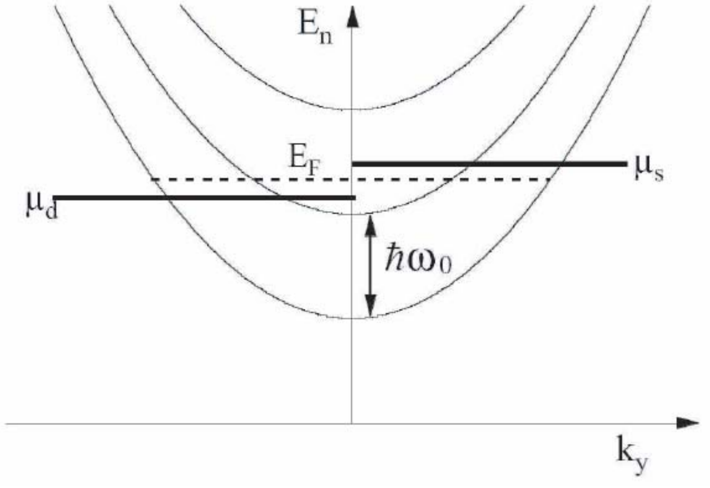
Electrons Confined in a Constriction

$$U \;=\; 1/2m^*\omega_0^2 y^2$$

$$\left[\frac{\hbar^2 k^2}{2m^*} + \frac{p_y^2}{2m^*} + \frac{1}{2}m^*\omega_0^2 y^2\right]\chi(y) = \chi(y)$$

... textbook ...

$$E(n,k) \;=\; \frac{\hbar^2 k^2}{2m^*} + \left(n + \frac{1}{2}\right)\hbar\omega_c, \quad n = 0, 1, 2, \ldots$$



$$V_{sd} = (\mu_s - \mu_d)/e$$

$$v(n,k) = \frac{1}{\hbar}\frac{\partial E(n,k)}{\partial k} = \frac{\hbar k}{m^*}$$

as for free electrons

Transport through a Constriction

$$I = e\sum_{n=1}^{N}\int_{\mu_d}^{\mu_s} dE\frac{1}{2}\rho_n(E)v_n(E)T_n(E)$$

$\rho_n(E) = 2/\pi(dE_n/dk_x)^{-1}$ 1D density of states

$T_n(E)$ transmission probability of the n^{th} subband

$$\begin{aligned}
I &= e\sum_{n=1}^{N}\int_{\mu_d}^{\mu_s} dE\frac{1}{2}\frac{2}{\pi}\left(\frac{\partial E_n}{\partial k_x}\right)^{-1}\frac{1}{\hbar}\frac{\partial E_n}{\partial k_x}T_n(E_F) \\
&= \frac{2e}{h}\sum_{n=1}^{N}T_n(E_F)\int_{\mu_d}^{\mu_s} dE \\
&= \frac{2e}{h}\sum_{n=1}^{N}T_n(E_F)eV_{sd}.
\end{aligned}$$

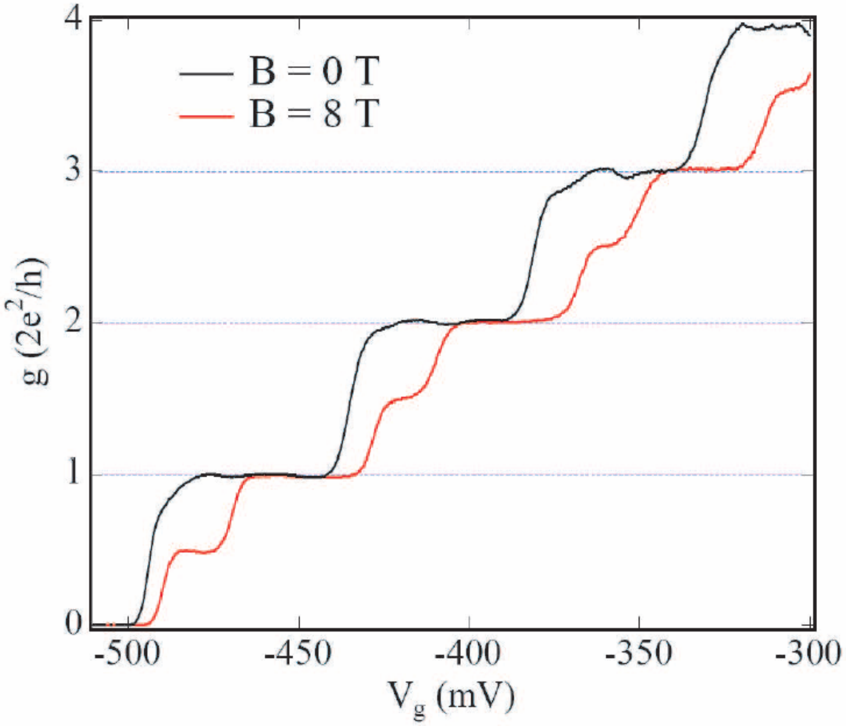
QUANTIZED!!

Transport through a Constriction: Conductance Quantization

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} T_n(E_F)$$

$$\sum_{n=1}^{N} T_n(E_F) = 1$$

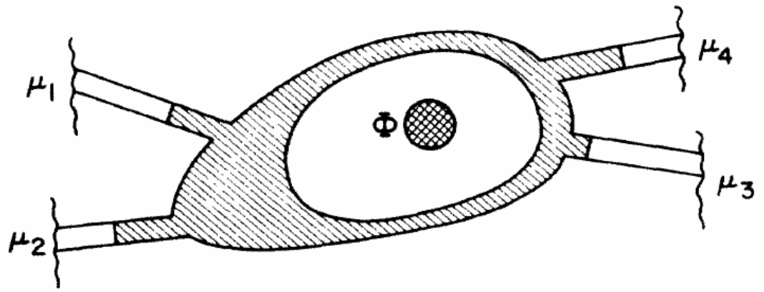
$$G = \frac{2e^2}{h} N$$



Landau-Büttiker Formalism

matrix $\mathbf{t}$

$$T_n(E_F) = \sum_{m=1}^{N} |t_{mn}|^2 \qquad \textit{two-terminal}$$



$$T_{\alpha\to\beta} = \sum_{n=1}^{N_\alpha}\sum_{m=1}^{N_\beta} |t_{\beta\alpha,mn}|^2 \ \textit{multi terminal}$$

greek: leads
roman: modes

$t_{\beta\alpha,mn}$ transmission probability amplitude from mode n in lead α to mode m in lead β

$$G = \frac{2e^2}{h}\sum_{n=1}^{N} T_n(E_F) = \frac{2e^2}{h}\sum_{n,m=1}^{N} |t_{mn}|^2 = \frac{2e^2}{h}\,\mathrm{Tr}\,\mathbf{t}\mathbf{t}^\dagger$$