# Introduction to Physics I

For Biologists, Geoscientists, &
Pharmaceutical Scientists

L16, 10.11.2017

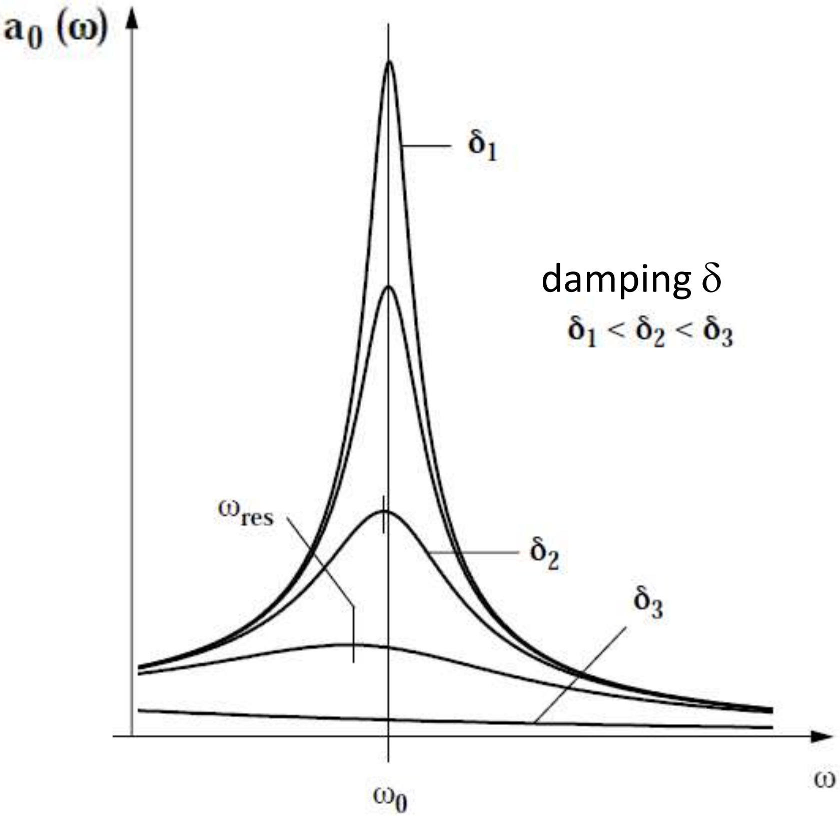
# driven oscillator


Tipler

# driven oscillator: resonance

***effect of damping***

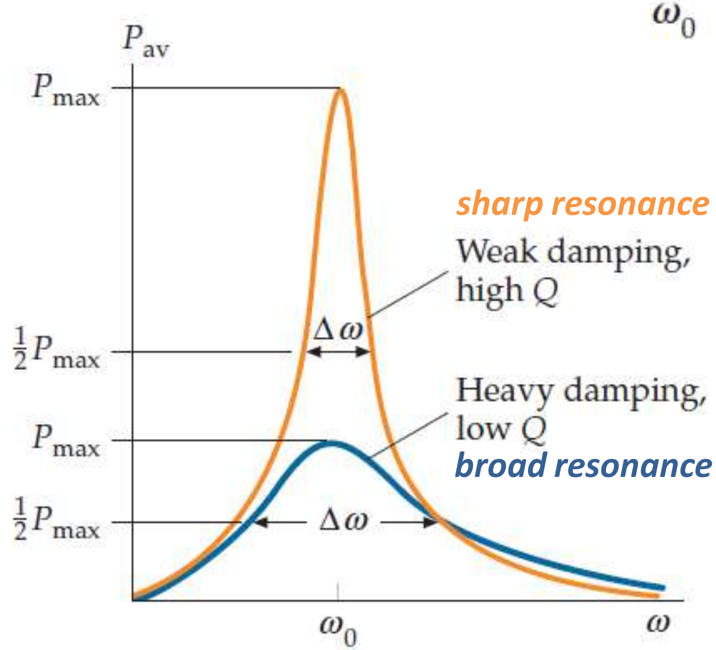
amplitude




damping δ

***average power delivered to an oscillator as a function of the driving frequency for two different values of damping***

Q: quality factor $\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$




***sharp resonance***

***broad resonance***

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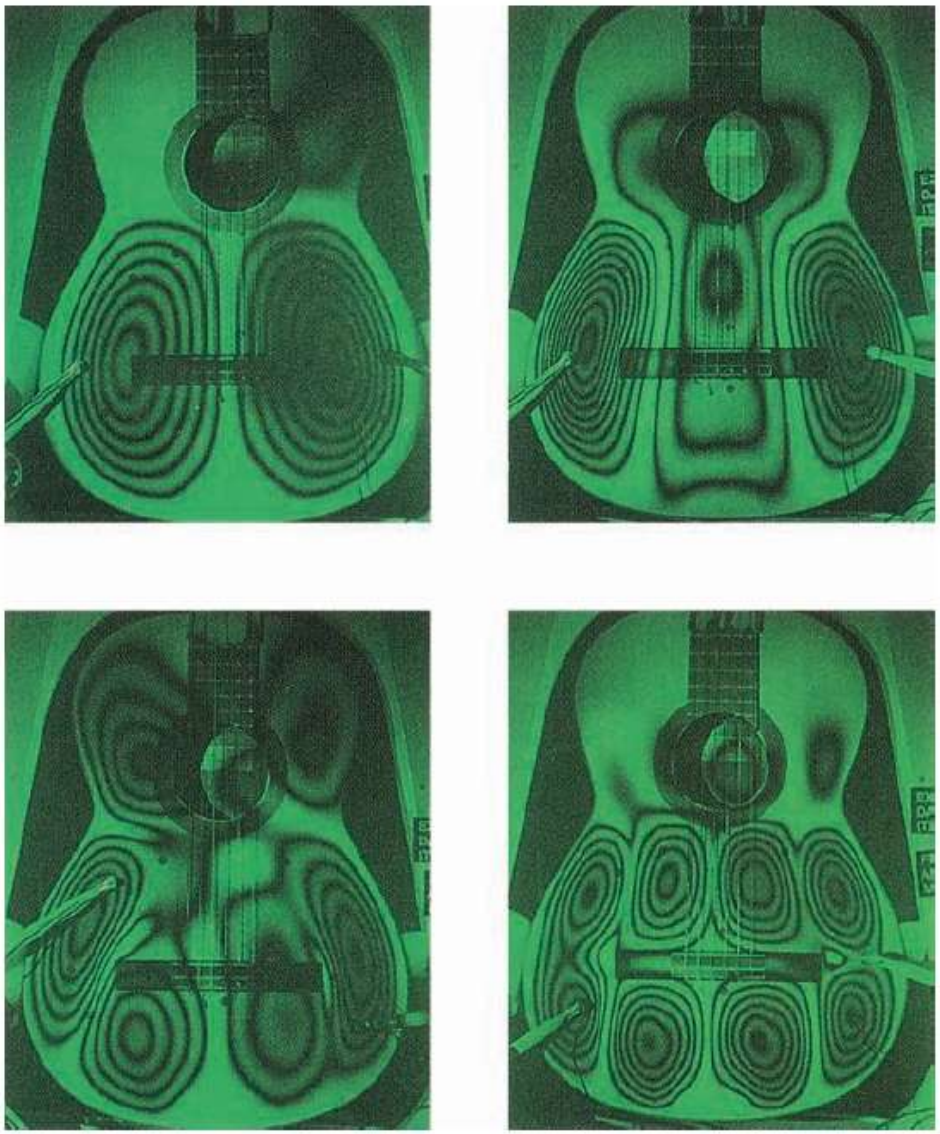
# resonances



Extended objects have more than one resonance frequency. When plucked, a guitar string transmits its energy to the body of the guitar. The body's oscillations, coupled to those of the air mass it encloses, produce the resonance patterns shown. *(Royal Swedish Academy of Music.)*

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# resonances

Massive damped oscillators were attached under the walkway shortly after this suspension bridge opened. The oscillators were put there to prevent the excessive swaying that was driven by lateral forces exerted by the footsteps of the walkers. *(Alamy.)*

https://www.youtube.com/watch?v=eAXVa__XWZ8

**London Millennium footbridge** (June 2000)

- 2000 people on bridge at same time
- walking*: lateral force component*, typ. freq. 1Hz
- 2 lowest natural freq. of sideways motion (144m-long center span): f=0.5 Hz, f2=1.0 Hz

⇨ **resonance easily driven**

- enhanced by natural behavior of people: lateral motion compensation by *synchronizing* walk: reinforced resonance

**Tacoma bridge**

https://www.youtube.com/watch?v=lXyG68_caV4

animation: aeroelastic flutter (wiki)

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# superposition of oscillations

***constructive interference***

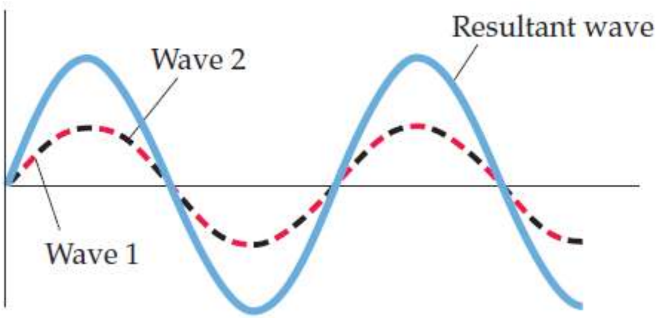


**FIGURE 16-5** Constructive interference. If two harmonic waves of the same frequency are in phase, the amplitude of the resultant wave is the sum of the amplitudes of the individual waves. Waves 1 and 2 are identical, so they appear as a single harmonic wave. Wave 1 is shown as a red dashed curve and Wave 2 is shown as a black dashed curve.

***destructive interference***

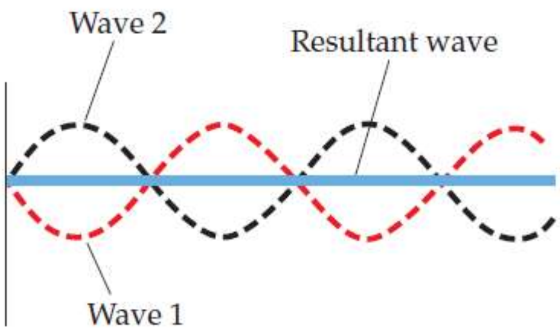


**FIGURE 16-6** Destructive interference. If two harmonic waves of the same frequency differ in phase by 180°, the amplitude of the resultant wave is the difference between the amplitudes of the individual waves. If the original waves have equal amplitudes, they cancel completely.

When two or more waves overlap, the resultant wave is the algebraic sum of the individual waves.

PRINCIPLE OF SUPERPOSITION

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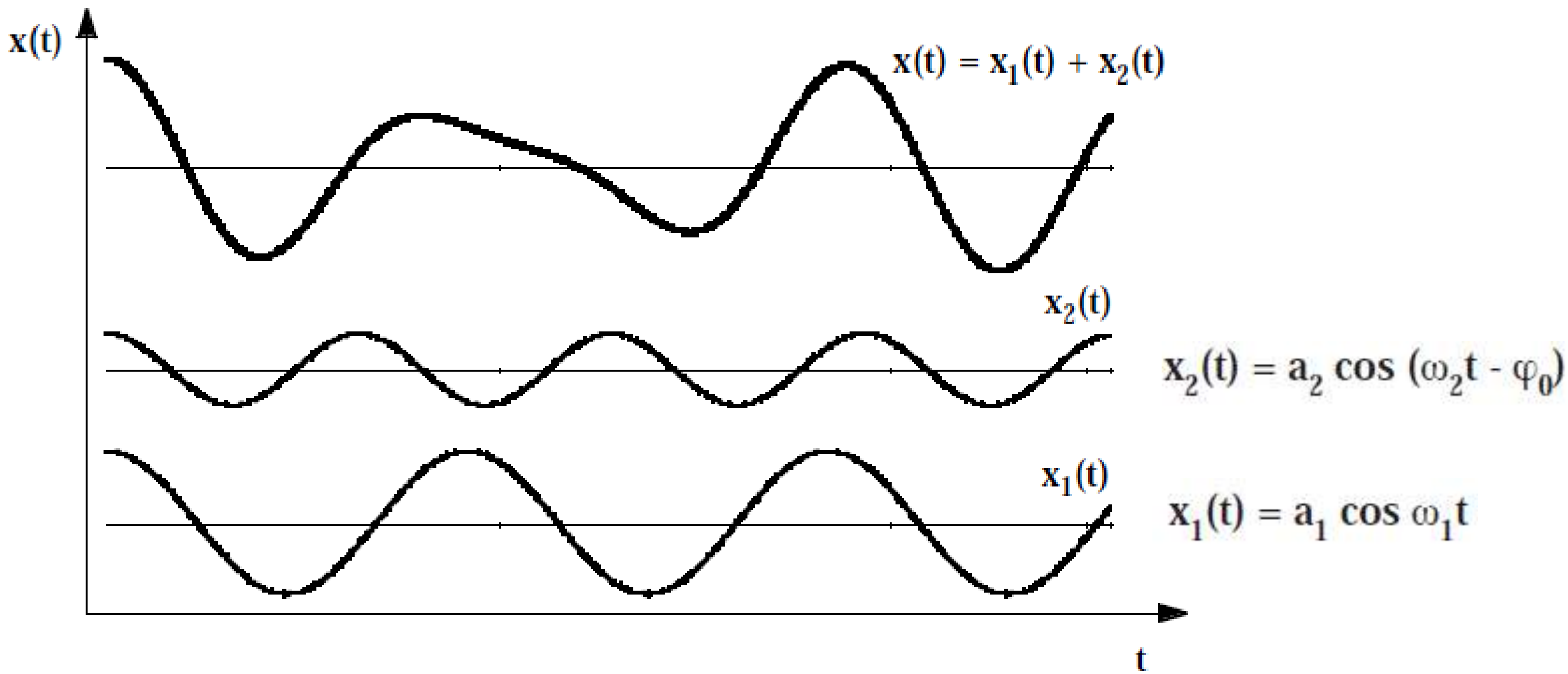
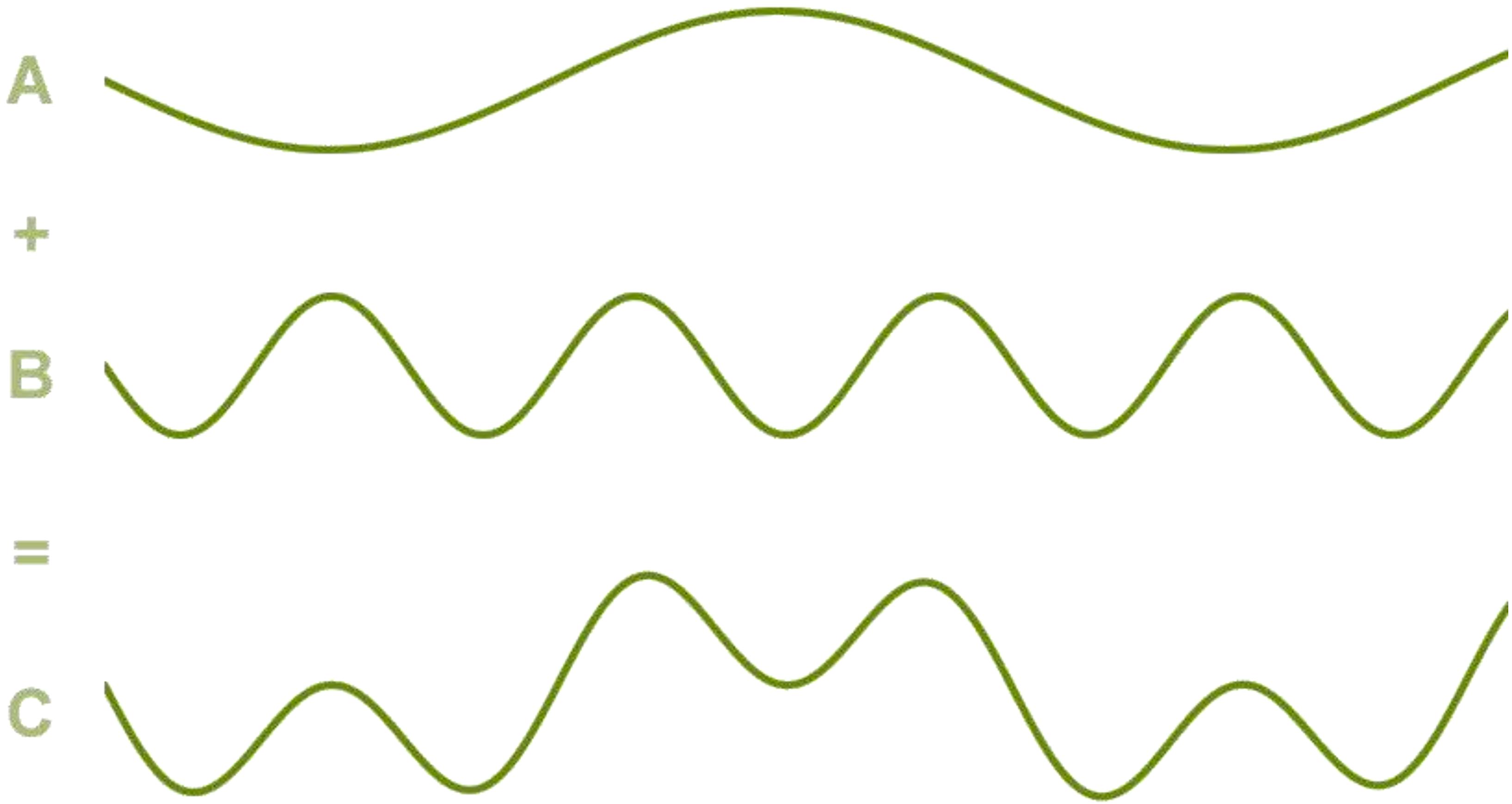
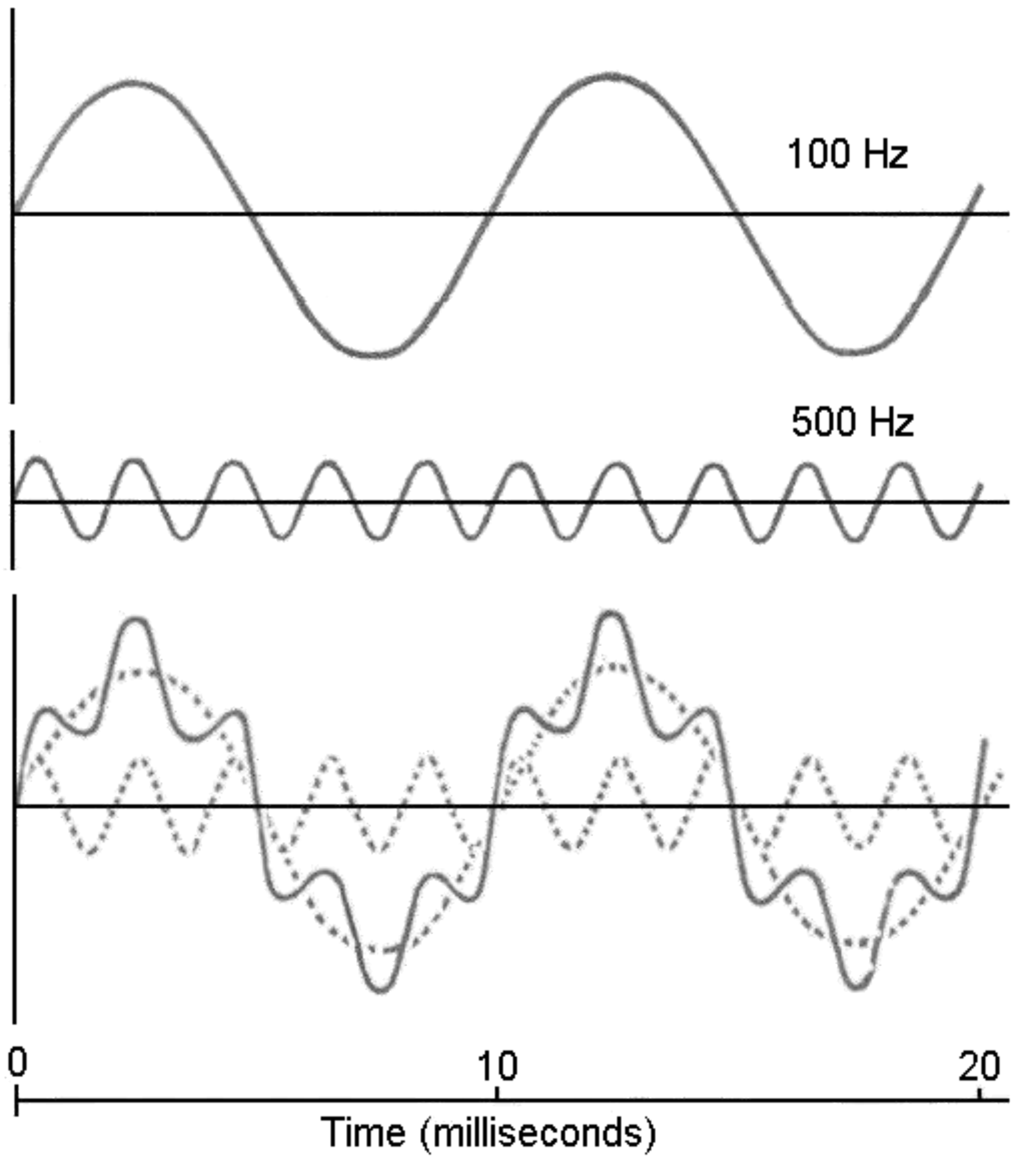
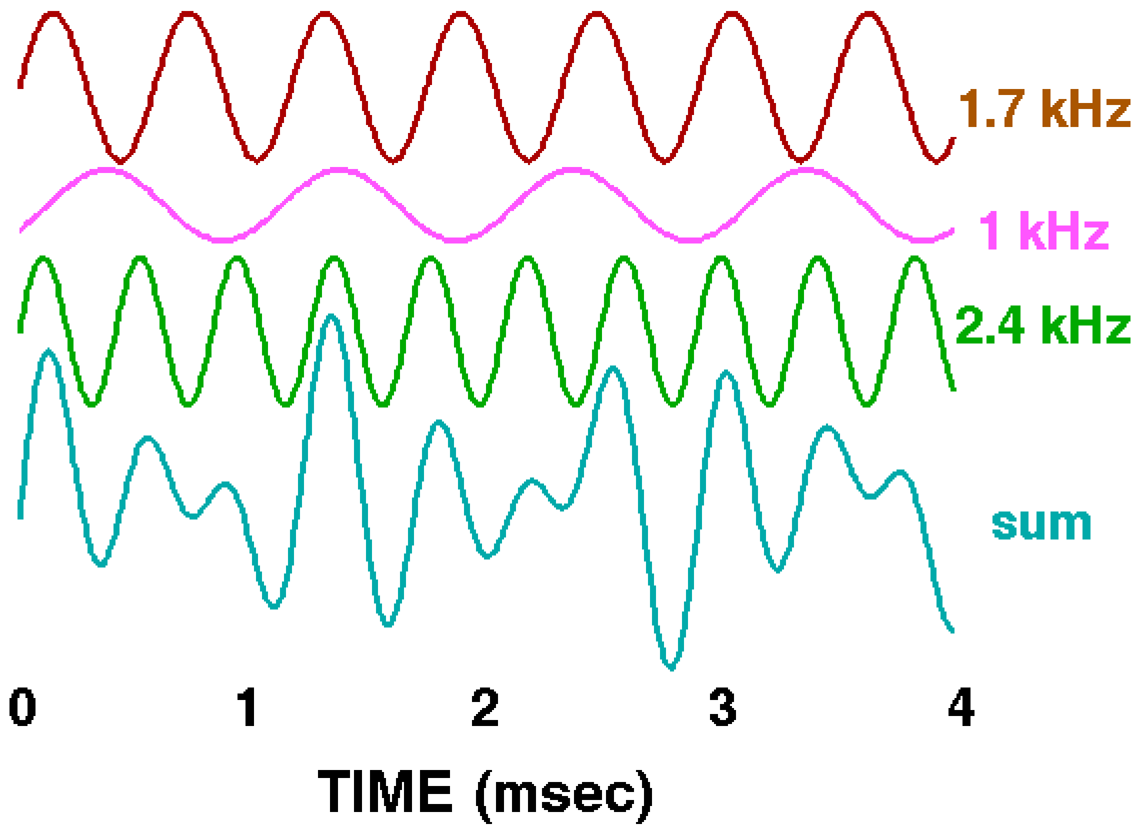
# superposition of oscillations

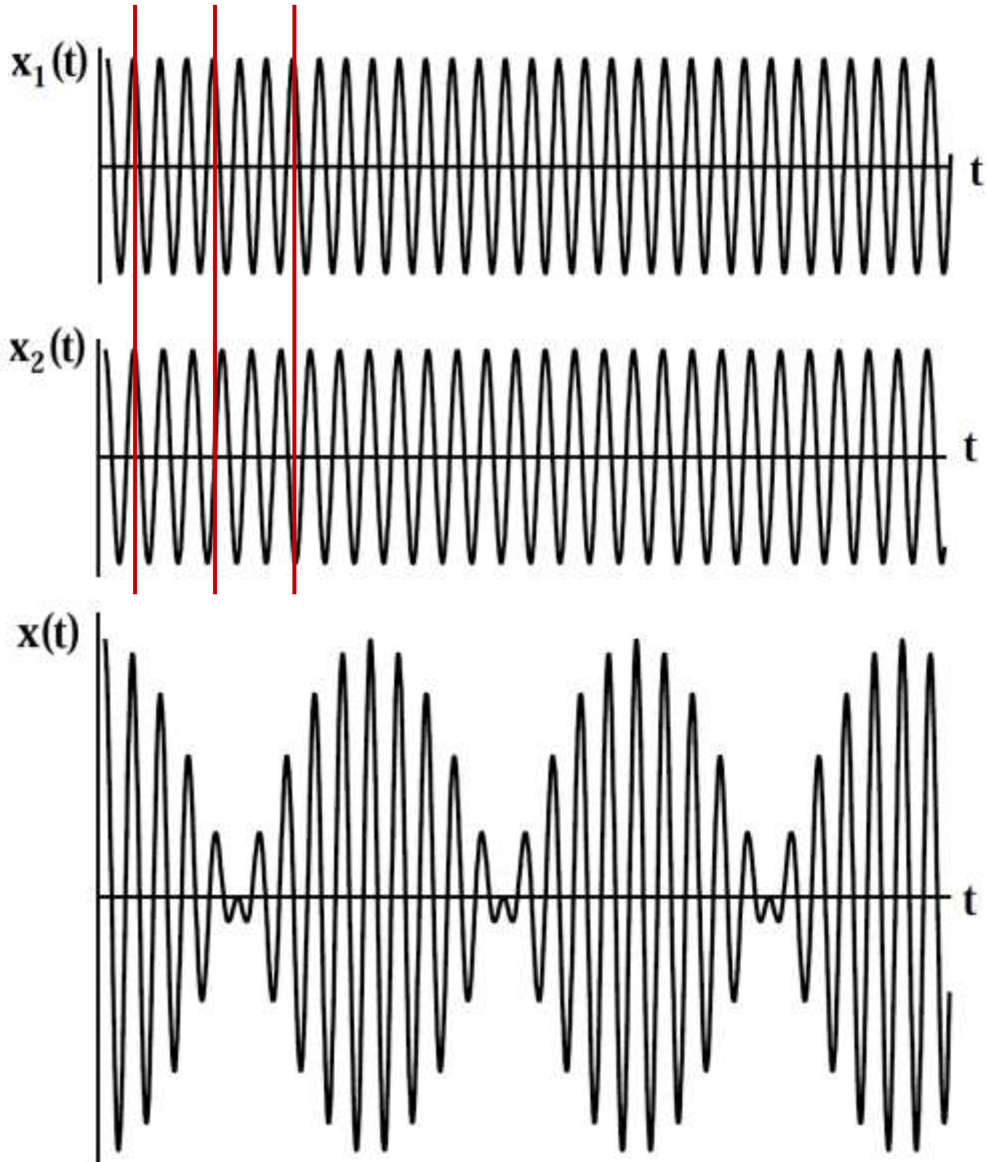



A
+
B
=
C


100 Hz
500 Hz
0
10
20
Time (milliseconds)

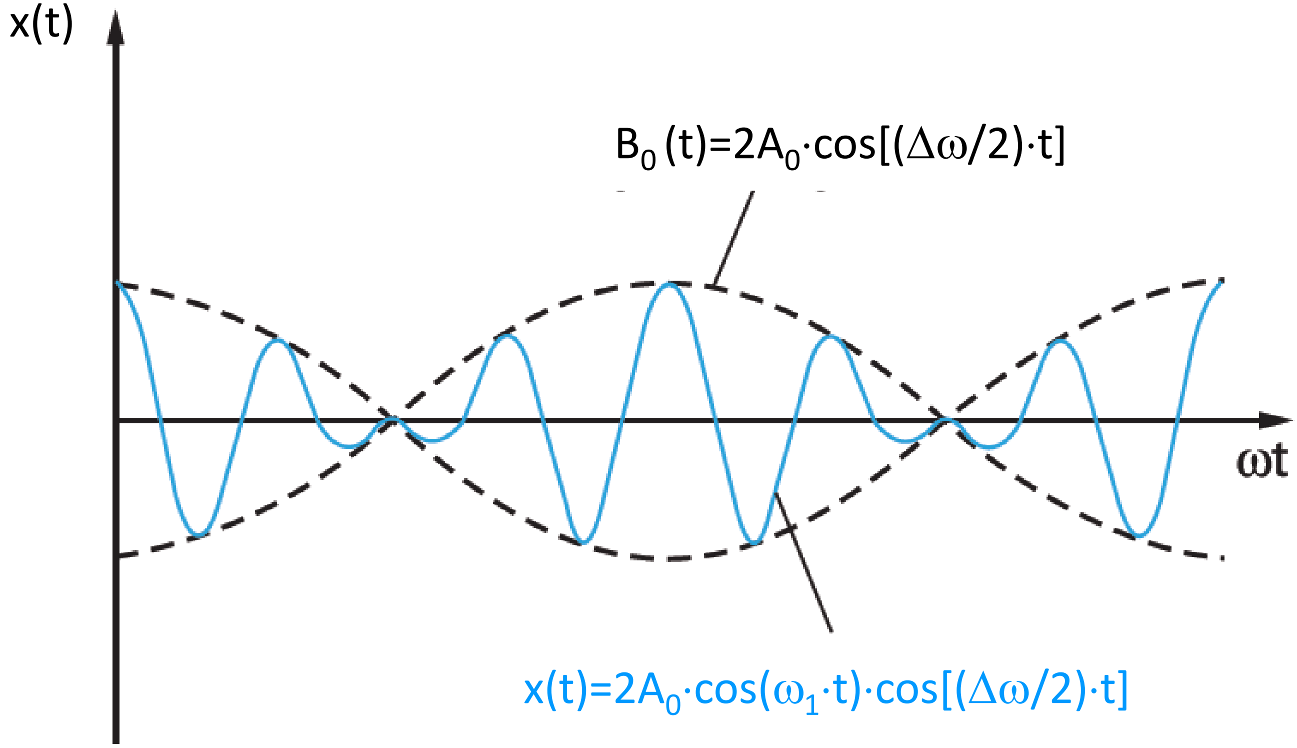

1.7 kHz
1 kHz
2.4 kHz
sum
0
1
2
3
4
TIME (msec)

# superposition of oscillations: beat




high freq. &
low freq.

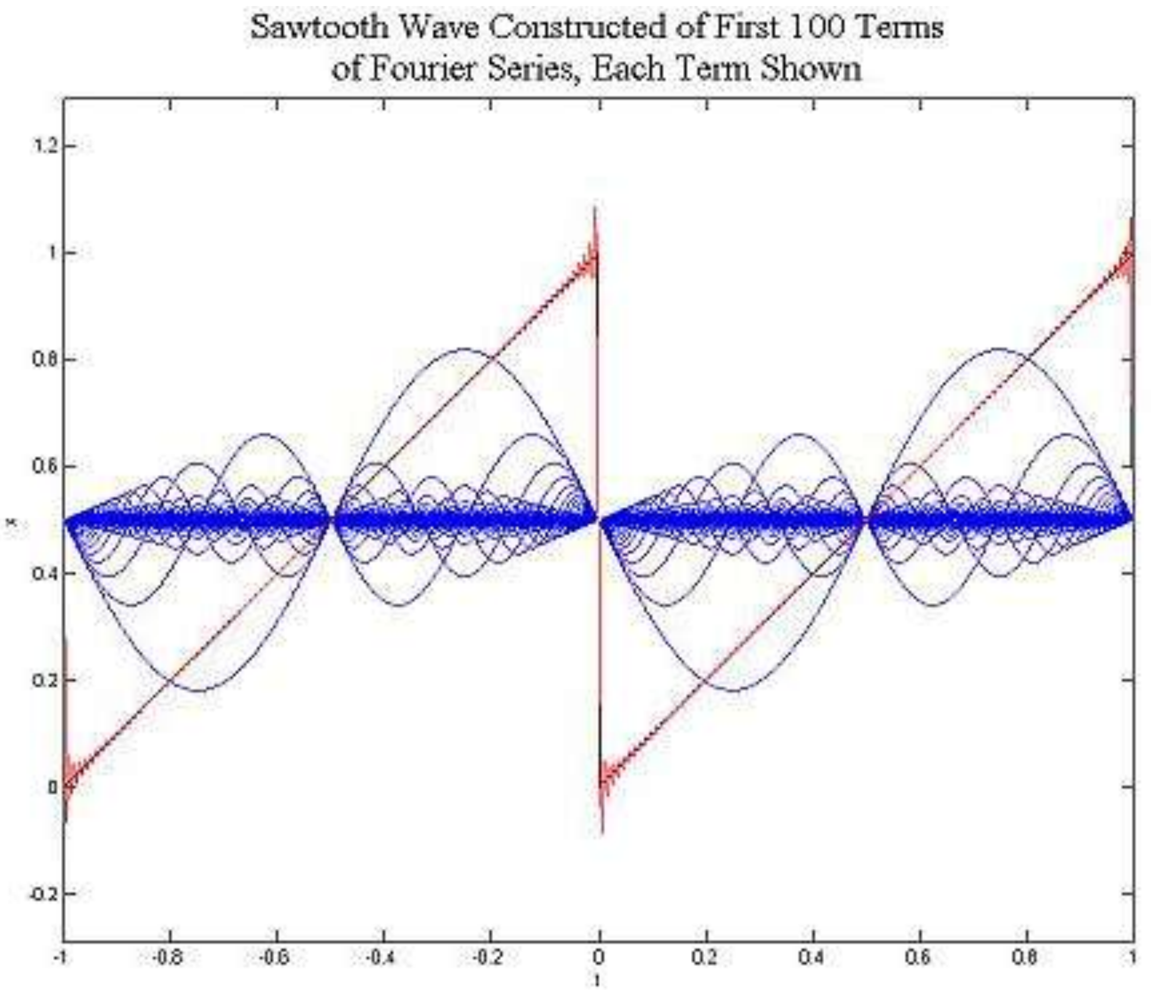
# superposition of oscillations: beat




$x(t)=B_0(t)\cdot\cos(\omega_1\cdot t)$

***Trautwein***

# Fourier series



# Fourier series & harmonic analysis

A (440Hz)

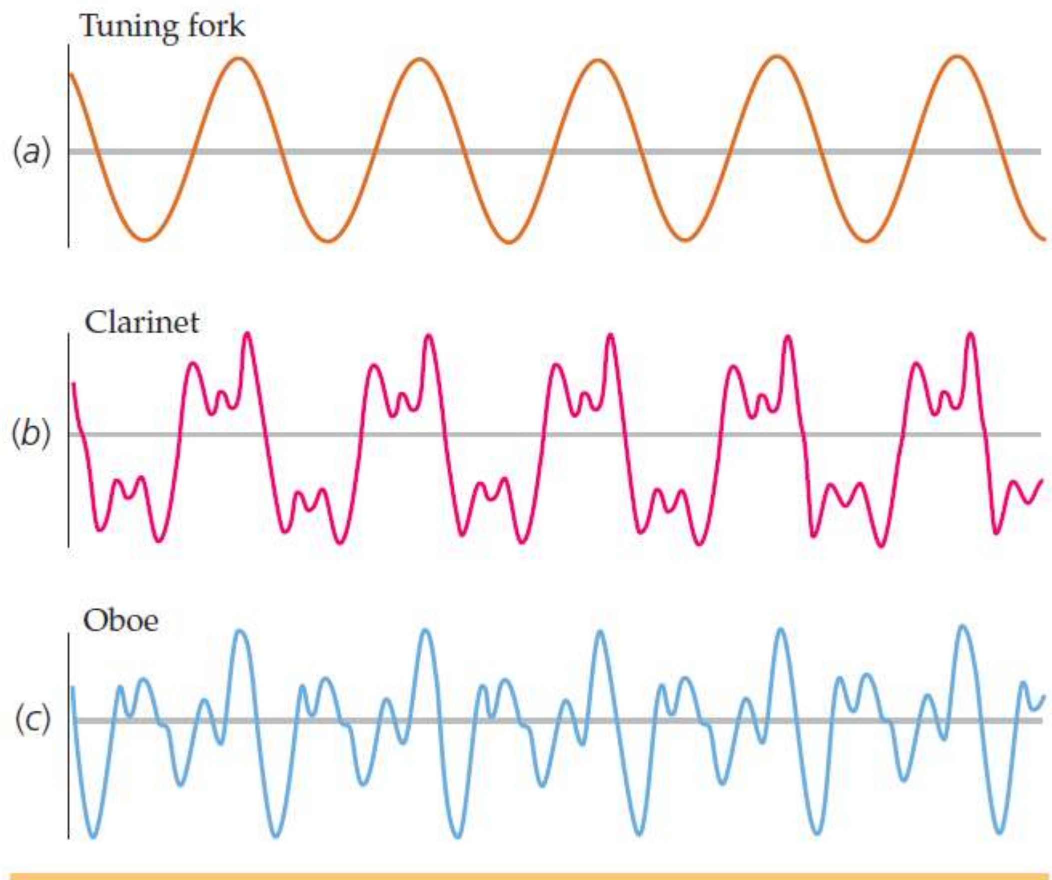



**FIGURE 16-25** Waveforms of (*a*) a tuning fork, (*b*) a clarinet, and (*c*) an oboe, each at a fundamental frequency of 440 Hz and at approximately the same intensity.

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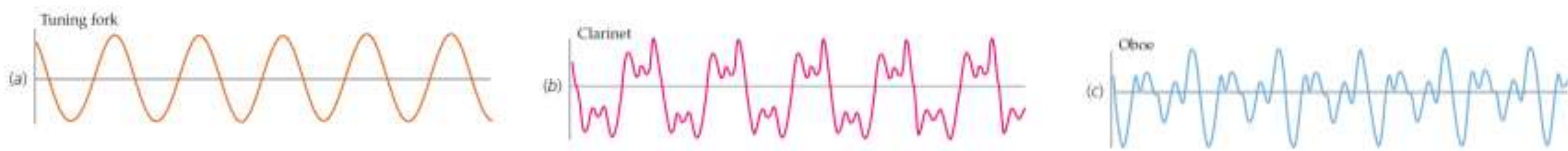
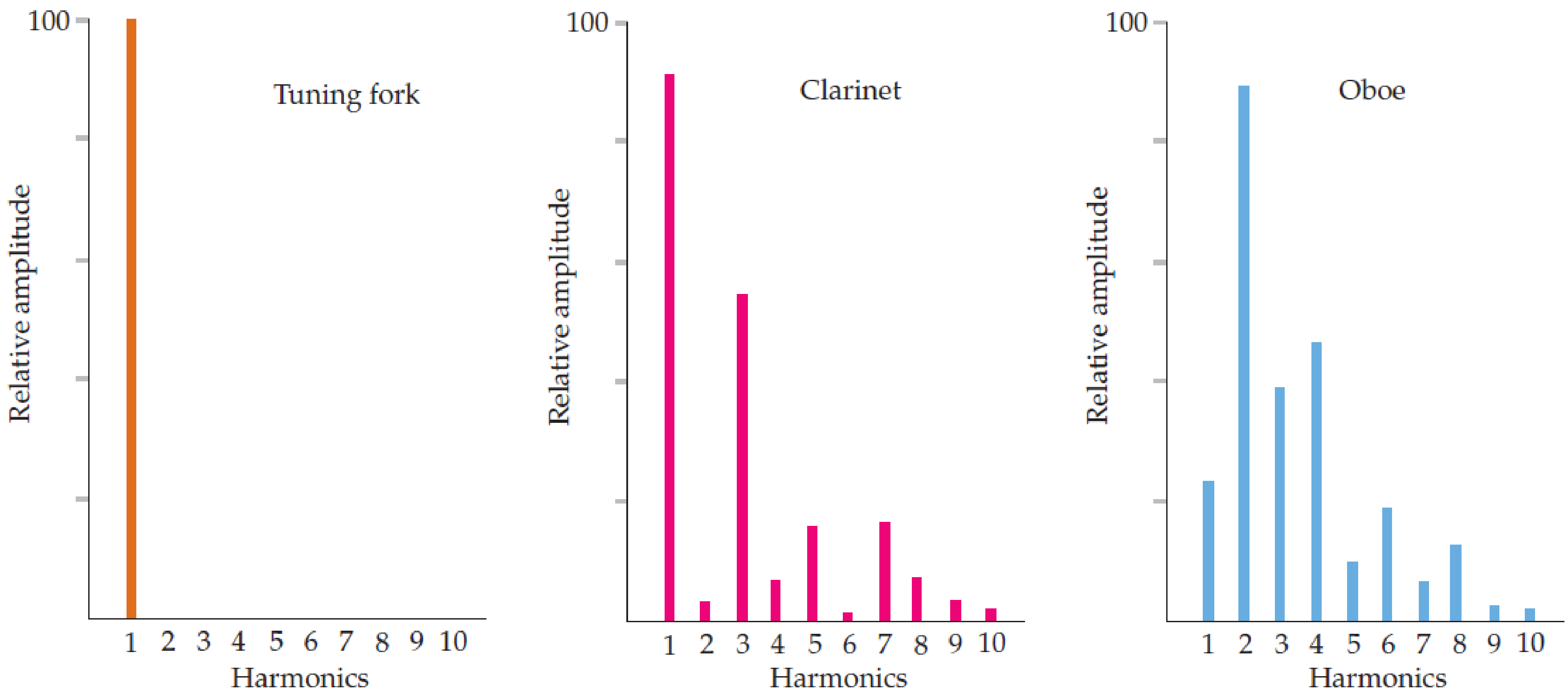
# Fourier series & harmonic analysis




***Relative amplitudes of the harmonics***

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# Fourier series & superposition: square wave

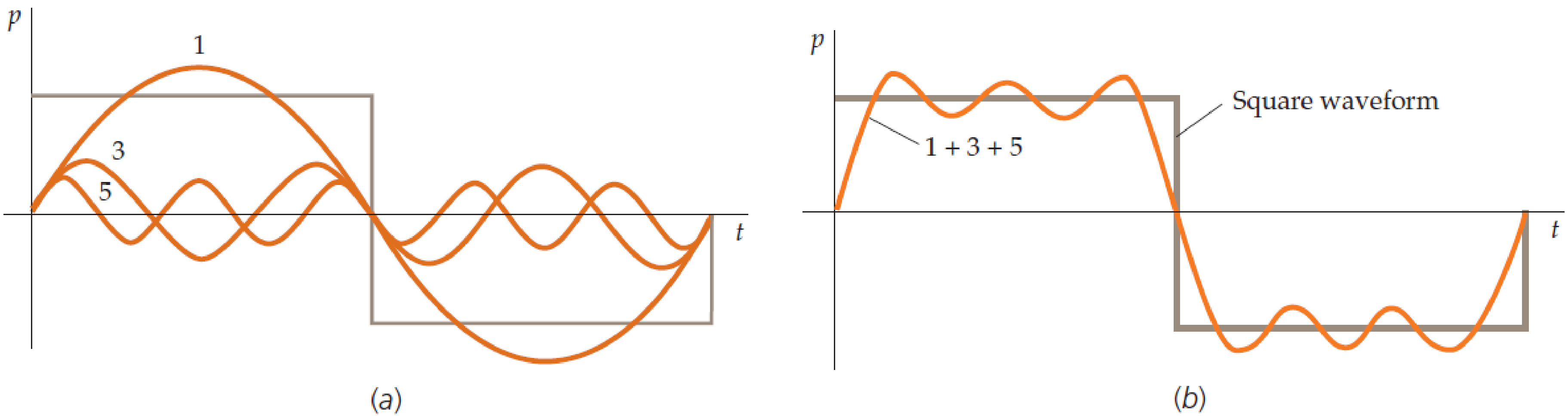


**FIGURE 16-27** (a) The first three odd harmonics used to synthesize a square wave. (*b*) The approximation of a square wave that results from summing the first three odd harmonics in (*a*).

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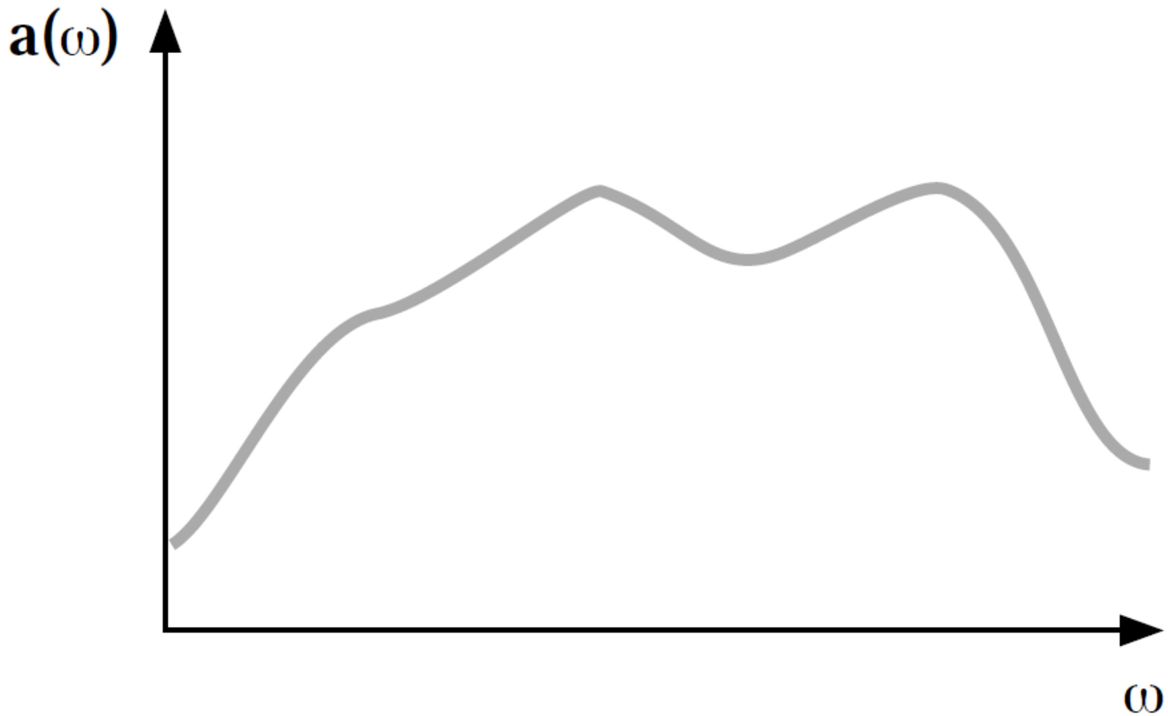
# Frequency spectrum of a bang



# coupled oscillators

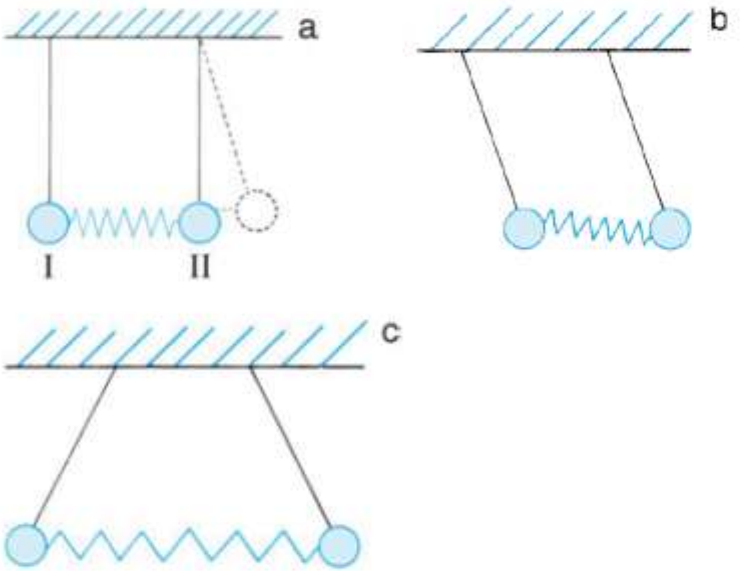



**Abb. 6.10** Zwei gekoppelte Fadenpendel (a) und ihre beiden Eigenschwingungen: (b) gleichphasig, (c) gegenphasig.

***trading energy between pendulum I & II result in alternating displacement***

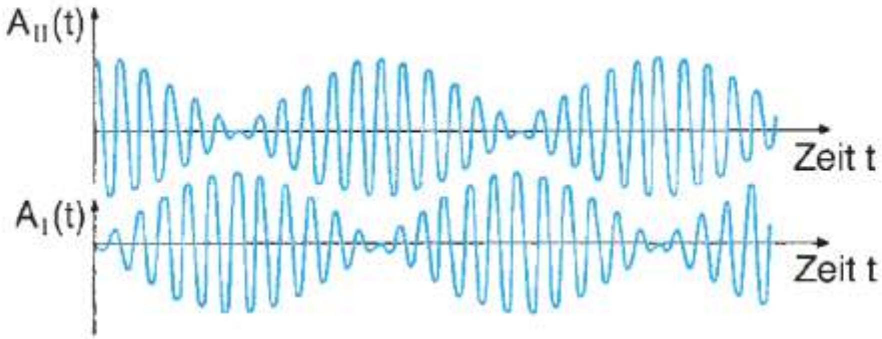



**Abb. 6.11** Schwingungen $A_I(t)$ und $A_{II}(t)$ der beiden gekoppelten Pendel I und II in Abb. 6.10.

***Trautwein***

# coupled oscillators, towards waves

***fundamental & higher order vibrations***

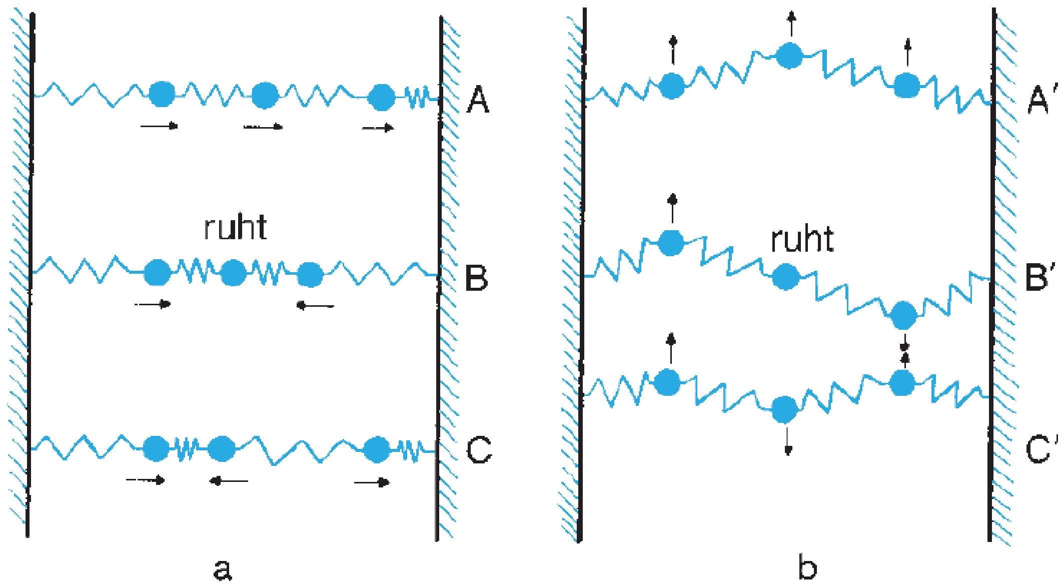
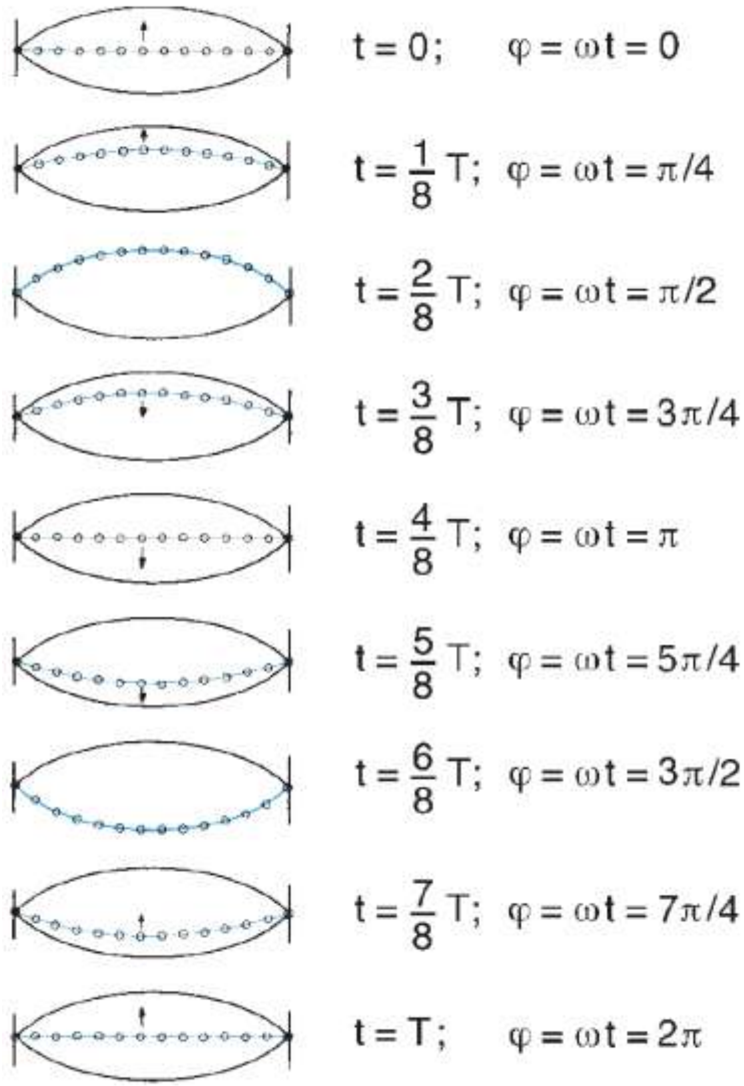



**Abb. 6.12** Grundschwingungen (*A*, *A*′), erste Oberschwingungen (*B*, *B*′) und zweite Oberschwingungen (*C*, *C*′) eines Systems aus drei gekoppelten Federpendeln: (a) Longitudinalschwingungen, (b) Transversalschwingungen.

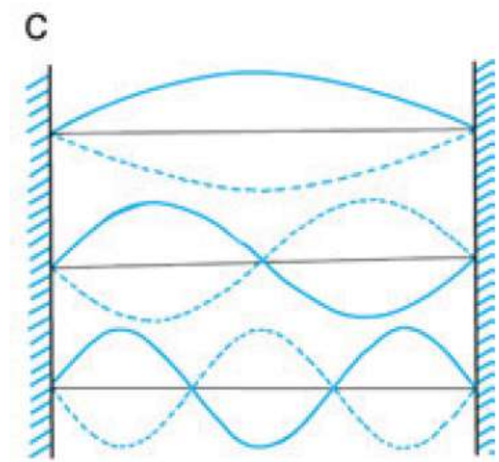
*Trautwein*

# coupled oscillators, towards waves

***fundamental mode transverse wave***
*e.g. atomic chain*




***fundamental mode and higher order modes (1 and 2)***




***Trautwein***

# driven oscillations and resonance

(slide) - to counter damping and keep system oscillate: drive / pump the oscillator

eq. of motion ; $\sum F_x = m \cdot a_x$

$$m \cdot \frac{d^2x}{dt^2} = - b\frac{dx}{dt} - kx + F_0 \cos(\omega t)$$

friction, restoring force (spring), external, driving force

and

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega_0^2 \cdot x = F_0 \cos(\omega t) \quad (1) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

## Qualitative discussion of the solution to (1)

- transient part, (with $x(t) = A_0 \cdot e^{-(b/2m)t} \cdot \cos(\omega' t + \varphi_0)$

$$\omega' = \omega_0 \sqrt{1 + \left(\frac{b}{2m\omega_0}\right)^2}$$

→ becomes negligible due to $e^{-b/2m \cdot t}$ factor ; ignore this part

- steady-state part for a driven oscillator

$$x = A \cdot \cos(\omega t - \varphi) \quad (2)$$

position

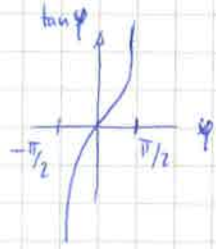
same angular freq. as driving force $F_0 = \cos(\omega t)$, but with a phase difference $\varphi$, oscillator lags behind driving force (damping $\varphi \propto b$)

$$A = \frac{F_0}{\left(m^2(\omega_0^2 - \omega^2)^2 + b^2 \cdot \omega^2\right)^{1/2}} \quad (3)$$

amplitude

$$\tan\varphi = \frac{b\,\omega}{m(\omega_0^2 - \omega^2)} \quad (4)$$

phase constant




- if $\omega \to 0$, $\varphi \to 0$ ; if $b \to 0$, no damping, $\varphi \to 0$

at $\omega = \omega_0$, $\varphi = \pi/2$ (90°) ; note: if $\omega \to \omega_0$ and $b \to 0$, $A \to \infty$

resonance

for $\omega \gg \omega_0$, $\varphi \to \pi$

velocity of driven oscillator:

$$v_x = \frac{dx}{dt} = -\omega A \cdot \sin(\omega t - \varphi)$$

at resonance, $\varphi = \pi/2$, hence $\sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t)$

and

$$v_x = \omega \cdot A \cdot \cos(\omega t)$$

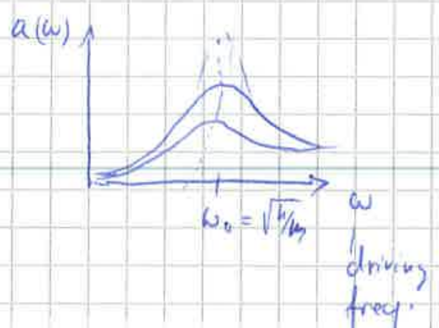
at resonance, object moves in direction of driving force $F_0 \cdot \cos(\omega t)$

(slide) - amplitude $A(\omega)$
- quality factor

(slide) . various resonance frequencies for guitar
note: calculation so far for 1 mass/resonator oscillator or 1 elementary oscillator

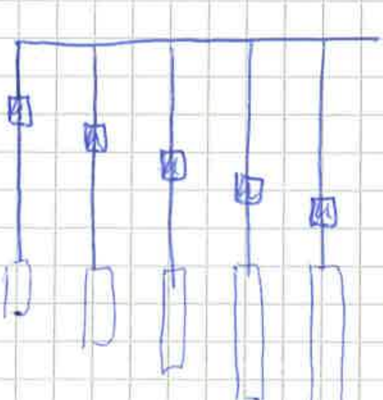
exp (108-8) driven oscillating mass (Federpendel, erzwungen)

a) without damping, show resonance
b) with damping





exp (108-10) row of pendulum (Pendelreihe)



- driven system
- $\omega_i$ different for each pendulum (resonance)
- slow driving, then increase speed: go through resonance for each pendulum

(damping adjustment)

(video) ~~Tacoma bridge~~

415 7"
416

(slide) London millenium footbridge
+ video

(Video) Tacoma bridge

15" : barge with concrete
49" : opening
1'18 : wind / gale : triggers resonance !
2' : expert / Prof. Farquharson trying to get dog...
walking the middle line
3'05 : collapse
3'47 : guys running

# Superposition of oscillations

(slides) examples :
- constructive / destructive interferen; superposition
- examples

- superposition of harmonic oscillation results in oscillation, but not necessarily harmonic
(i.e. restoring force $F_{1+2} \propto -x$ not verified any more although $F_1 \propto -x$, $F_2 \propto -x$ )

def: anharmonic oscillation : periodic oscillation that cannot be described by a single sin or cos function

- superposition of 2 harmonic oscillators with similar frequencies

%

$$x_1(t) = A_0 \cdot \cos(\omega_1 t)$$
$$x_2(t) = A_0 \cdot \cos(\omega_2 \cdot t)$$
$$x(t) = x_1(t) + x_2(t) = 2 A_0 \cdot \cos\left(\underbrace{\frac{\omega_1+\omega_2}{2}}_{\text{high freq.}} \cdot t\right) \cdot \cos\left(\underbrace{\frac{\omega_1-\omega_2}{2}}_{\text{low freq.}} \cdot t\right)$$

(slide) ⇒ beat (Schwebung) : amplitude modulation of high freq. oscillation $\left(\frac{\omega_1+\omega_2}{2}\right)$ at low freq. $\left(\frac{\omega_1-\omega_2}{2}\right)$

analysis : (%) p 8'

exp : organ pipes
(108-11)

a) show hear 1 column / pipe ; tune freq. by changing size
b) 2 pipes in tune
change 1 tube

exp tuning forks 1, 2 512 Hz
(108-11)

a) both : in phase, same freq.
b) add damping (paste) → beat
c) "transmit" / couple vibration from 1 to 2, position tuning fork boxes properly

strobe light ~ 50 Hz
→ see vibration "slower"
(superposition of fork vibration with light oscillation → beat pattern)

L15
L16 8'

## beat analysis :

$$x(t) = x_1(t) + x_2(t) = 2\,A_0 \cdot \cos\left(\frac{\omega_1+\omega_2}{2}t\right) \cdot \cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$\begin{pmatrix} x_1(t) = A_0 \cos(\omega_1 t) \\ x_2(t) = A_0 \cos(\omega_2 \cdot t) \end{pmatrix}$$

$\omega_1 \sim$ similar to $\omega_2$ $\Rightarrow$ $\omega_1 = \omega_2 + \Delta\omega$ , $\Delta\omega \ll \omega_1, \omega_2$

$$\omega_1 + \omega_2 = \omega_2 + \Delta\omega + \omega_2 = 2\omega_2 + \Delta\omega \simeq 2\omega_2 \simeq 2\omega_1$$

$$\omega_1 - \omega_2 = \Delta\omega$$

i.e. :

$$x(t) \cong 2 \cdot A_0 \cdot \cos(\omega_1 t) \cdot \cos\left(\frac{\Delta\omega}{2} \cdot t\right)$$

$$= 2 \cdot A_0 \cdot \cos\left(\frac{\Delta\omega}{2} \cdot t\right) \cdot \cos(\omega_1 t)$$

$$x(t) = B_0(t) \cdot \cos(\omega_1 t)$$

← same freq as fundamental freq. in system $\omega_1 \simeq \omega_2$ ($\omega_1 - \omega_2$ small)

$$B_0(t) = 2 \cdot A_0 \cdot \cos\left(\frac{\Delta\omega}{2} \cdot t\right)$$

(slide)

L15
L16 18''

exp : superposition of 2 oscillators on oscilloscope

500 Hz
500.02 Hz | listen to beat.

Fourier analysis : observed superp. of 2 harmonic oscill gave nonharmonic oscill

math/theory ⇒ || periodic oscillation with period T can be expressed by sum of harmonic oscillations:

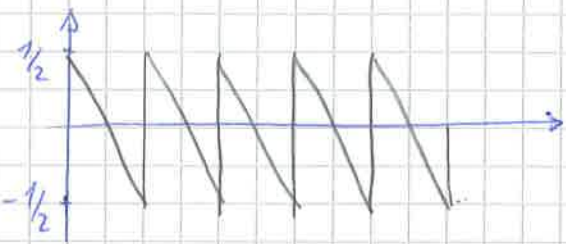
$$x(t) = x(t+T) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos(n \cdot \omega t) + \sum_{n=1}^{\infty} B_n \cdot \sin(n \cdot \omega \cdot t)$$

↑ period $\omega = \frac{2\pi}{T}$

(no demo here)

example

sawtooth oscillation




Fourier series (infinite sum)

$$x(t) = \underbrace{\frac{1}{\pi}}_{B_1} \cdot \sin(\omega t) + \underbrace{\frac{1}{2\pi}}_{B_2} \cdot \sin(2\omega t) + \frac{1}{3\pi} \sin(3\omega t) + \dots$$

$A_m = 0 \quad \forall m$

$B_m = \frac{1}{m \cdot \pi} \quad / \quad \xrightarrow[m \to \infty]{} 0$


frequency spectrum (Fourier or reciprocal space)

ω: fundamental frequency
nω: harmonics

(slide) . Fourier series ~~of a bang~~

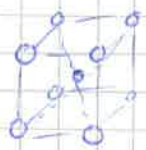
- sawtooth wave; 100 terms of Fourier series
- harmonic analysis: tuning fork, clarinet, oboe
- harmonic superposition: square wave
- bang, spectrum

( exp bang, tuning fork )

# Coupled oscillators : from oscillators to waves

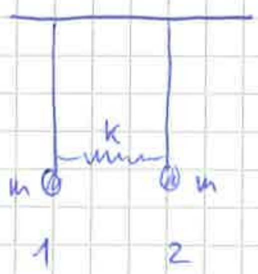
describe matter : molecule, graphene, crystals, ...

atoms are individual oscillators, coupled to each other by interaction forces
( bonds as "springs": harmonic approximation )



**exp** : coupled pendulum

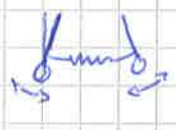
a)



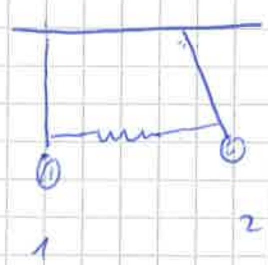

2 normal modes of oscillation :

- spring neither stretched nor compressed

- spring stretched and compressed



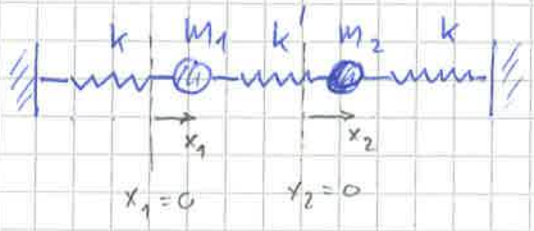
b)




coupled oscillation :
→ trading back & forth of energy & displacement between both masses
→ alternating between 2 moving, 1 at rest and 2 at rest, 1 moving.

(slide) modes, trading energy

coupled oscillators : eq of motion for 2 identical masses




$m_1 = m_2$

$x_1$ : extension of left spring

$-x_2$ : extension of right spring (compression)

$x_2 - x_1$ : extension of middle spring

$x_1 = x_2 = 0$ : springs not extended

$$m\cdot\ddot{x}_1 = -k\cdot x_1 + k'(x_2 - x_1)$$
$$m\cdot\ddot{x}_2 = -k'(x_2 - x_1) + k\cdot(-x_2)$$

coupled differential equations

$\omega_0 = \sqrt{\frac{k}{m}}$ , $\omega_k = \sqrt{\frac{k'}{m}}$

eig. freq. of uncoupled oscillators — "coupling" frequency

(also for oscill. in phase $(x_1 - x_2 = 0)$)

eq. of motion:

$$\ddot{x}_1 = -\frac{k}{m}\cdot x_1 - \frac{k'}{m}\cdot x_1 + \frac{k'}{m}x_2 = -(\omega_0^2+\omega_k^2)\cdot x_1 + \omega_k^2\cdot x_2$$

$$\ddot{x}_2 = -\frac{k'}{m}\cdot x_2 - \frac{k}{m}x_2 + \frac{k'}{m}\cdot x_1 = \omega_k^2\cdot x_1 - (\omega_0^2+\omega_k^2)\cdot x_2$$

relative coordinates: $q_1 = x_1 + x_2$, $q_2 = x_1 - x_2$

hence

$$\ddot{q}_1 = \ddot{x}_1 + \ddot{x}_2 = -\omega_0^2\cdot x_1 + \omega_0^2\cdot x_2 = -\omega_0^2\cdot q_1 = -\Omega_1^2\cdot q_1$$

$$\ddot{q}_2 = \ddot{x}_1 - \ddot{x}_2 = -(\omega_0^2+2\omega_k^2)\cdot x_1 + (\omega_0^2+2\omega_k^2)\cdot x_2$$

$$= -(\omega_0^2+2\omega_k^2)\cdot q_2$$

$$= -\Omega_2^2\cdot q_2$$

$\Omega_1, \Omega_2$: eigen-freq. of the fundamental oscillations of the system

$\Omega_1 = \omega_0 = \sqrt{\frac{k}{m}}$ ; masses oscillate in phase →→

$\Omega_2 = \sqrt{\omega_0^2 + 2\omega_k^2}$ : " " in anti phase →←

$\omega_k = \sqrt{\frac{k'}{m}}$

in phase: $x_1 = x_{1,0}\cdot\cos(\Omega_1 t)$
$x_2 = x_{2,0}\cdot\cos(\Omega_1 t)$

anti-phase: $x_1 = x_{1,0}\cdot\cos(\Omega_1 t)$
$x_2 = -x_{2,0}\cdot\cos(\Omega_1 t)$

NB: N coupled oscillators : N eigenfrequencies
for linear chain of coupled oscillators

towards waves :

nb masses increases → behavior of molecules, etc..
↑
atoms...

longitudinal waves :

transverse wave :

(slides) : 6.12 Trautwein
1) long. + transv. & higher order vibrations

in 3D : N coupled oscillators (atoms) → 3N normal modes
(eigen frequencies)
in solids : lattice vibrations
quantum description : phonons

2) fundamental transversal vibration for atomic chain :
Fig 6.13

+ higher order modes, Fig 6.14 c